Degrees-of-Freedom for the 4-User SISO Interference Channel with Improper Signaling

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***Abstract*—It has been recently shown that for the 3-user single-input single-output (SISO) interference channel with constant channel coefficients, a maximum of 1.2 degrees-of-freedom (DoF) are achievable using linear interference alignment schemes when improper (a.k.a. asymmetric) Gaussian signaling is applied. In this paper, we study the 4-user SISO interference channel and provide inner and outer bounds for the total number of DoF achievable for this channel. We prove that at least $\frac{4}{3}$ DoF are achievable for the 4-user channel using also linear interference alignment techniques and improper signaling. A simple converse proof shows that no more than $\frac{8}{5}$ DoF are achievable for this scheme. Simulation results seem to indicate that the inner bound is in fact tight for this channel, and serve to illustrate the sum-rate improvement with respect to time division multiple access (TDMA) techniques.**

***Index Terms*—Degrees-of-freedom (DoF), interference channel, interference alignment, improper signaling, asymmetric signaling**

I. Introduction

Interference alignment (IA) has emerged as a novel technique to deal with interference in wireless networks. Using this approach, the interference coming from all users is confined into a subspace that does not overlap with the signal subspace, thus yielding an interference-free communication link for the desired signal [1]. Interference alignment techniques using the space, time and/or frequency dimensions have been proven to be essential to achieve the maximum degrees-of-freedom DoF in many interference channel scenarios. For instance, in [1] it is shown that for the K-user single-input-single-output (SISO) interference channel with time-varying coefficients, $\frac{K}{2}$ DoF are achievable using IA. The situation when the SISO channel coefficients remain constant is much more complicated, and it was long conjectured that only 1 DoF was achievable for this scenario [2].

Recently, Cadambe, Jafar and Wang disproved this conjecture in [3] and showed that the 3-user SISO interference channel has 1.2 DoF (for generic channels drawn from a continuous distribution). Along with the use of IA, another key ingredient to achieve 1.2 DoF for this scenario is the use of improper or noncircular complex Gaussian inputs (also referred to as asymmetric signaling in [3]). The use of improper Gaussian signaling has recently been shown to enlarge the achievable rate region for the 2-user SISO interference channel even when the interference is treated as noise [4], [5].

In this work, we study the achievable DoF for the 4-user SISO interference channel with constant channel coefficients. We focus also on linear interference alignment schemes based on asymmetric complex signaling, and provide inner and outer bounds. Specifically, our main result shows that at least $\frac{4}{3}$ DoF are achievable for this channel. More concretely, each user is able to transmit one complex data stream in 3 symbol extensions.[1] In the proof of our results we exploit some important differences between the 3-user and the 4-user channels. In particular, a fundamental limitation for the 3-user channel, which states that any given vector cannot align with the interference at more than one undesired receiver [3], is overcome for the 4-user channel.

By means of simulation results, in the paper we also compare the average sum-rate performance of the proposed signaling scheme with respect to conventional TDMA techniques. The analysis of the IA solutions obtained using an alternating minimization algorithm [7] allows us to conjecture that the DoF inner bound is tight and, in consequence, that the 4-user SISO interference channel with constant channel coefficients has $\frac{4}{3}$ DoF. We also study the total number of DoF when the number of users, K, grows.

The rest of the paper is organized as follows: Section II describes the system model. In Section III we highlight a fundamental difference between the 3-user and the 4-user channel that will be used in our achievability proof. The main contribution of this work is described in Section IV, where the inner and the outer bounds are derived. In Section V we present some numerical examples, including achievability results for different number of users. Finally, we conclude the paper in Section VI.

II. System Model

We consider a 4-user complex Gaussian interference channel with single-antenna users, as depicted in Fig. 1. We assume that the channel coefficients remain constant during the transmission of a whole data block and are perfectly known

[1] It is interesting to mention that, as shown in [6], the 4-user SIMO channel when each receiver is equipped with 2 antennas and the channel coefficients are *time-varying* has $\frac{8}{3}$ DoF (i.e., twice the total DoF with constant channels and single antenna receivers, as proved in this paper). Furthermore, the achievability result in [6] requires $3(n+1)^8$ symbol extensions for any $n \in \mathbb{N}$, whereas we get $\frac{4}{3}$ DoF with only 3 symbol extensions and constant channels.

to all transmitters and receivers. Each transmitter-receiver pair communicates over the same frequency band and time slots, thus interfering with one another. With these considerations, the discrete-time signal received by the ith user, during the nth channel use can be expressed as

$$y_i[n] = \sum_{j=1}^{4} h_{ij} x_j[n] + r_i[n] \tag{1}$$

where $h_{ij} = |h_{ij}| e^{\jmath \phi_{ij}}$ is the complex channel coefficient between transmitter j and receiver i, $x_j[n]$ is the signal transmitted by user j in the nth time slot and $r_i[n]$ is the additive white Gaussian noise (AWGN) at receiver i, with zero mean and variance σ^2. Eq. (1) can be alternatively represented using only real variables as follows

$$\underbrace{\begin{bmatrix} \Re\{y_i[n]\} \\ \Im\{y_i[n]\} \end{bmatrix}}_{\mathbf{y}_i[n]} = \sum_{j=1}^{4} |h_{ij}| \underbrace{\begin{bmatrix} \cos(\phi_{ij}) & -\sin(\phi_{ij}) \\ \sin(\phi_{ij}) & \cos(\phi_{ij}) \end{bmatrix}}_{\mathbf{R}(\phi_{ij})} \cdot \underbrace{\begin{bmatrix} \Re\{x_j[n]\} \\ \Im\{x_j[n]\} \end{bmatrix}}_{\mathbf{x}_j[n]} + \underbrace{\begin{bmatrix} \Re\{r_i[n]\} \\ \Im\{r_i[n]\} \end{bmatrix}}_{\mathbf{r}_i[n]}, \tag{2}$$

where $\Re\{\cdot\}$ and $\Im\{\cdot\}$ denote the real and imaginary parts, respectively. $\mathbf{R}(\phi)$ is a rotation matrix, which has the following properties

$$\begin{aligned} &\mathbf{R}(\phi)^{-1} = \mathbf{R}(-\phi) \ , \\ &\mathbf{R}(\theta)\mathbf{R}(\phi) = \mathbf{R}(\phi)\mathbf{R}(\theta) = \mathbf{R}(\phi+\theta) \ . \end{aligned} \tag{3}$$

Considering S symbol extensions, the signal received by the ith user is then given by

$$\underbrace{\begin{bmatrix} \Re\{y_i[Sn]\} \\ \Im\{y_i[Sn]\} \\ \vdots \\ \Re\{y_i[S(n+1)-1]\} \\ \Im\{y_i[S(n+1)-1]\} \end{bmatrix}}_{\bar{\mathbf{y}}_i[n]} = \sum_{j=1}^{4} |h_{ij}| \underbrace{[\mathbf{I}_S \otimes \mathbf{R}(\phi_{ij})]}_{\bar{\mathbf{R}}(\phi_{ij})} \cdot \underbrace{\begin{bmatrix} \Re\{x_j[Sn]\} \\ \Im\{x_j[Sn]\} \\ \vdots \\ \Re\{x_j[S(n+1)-1]\} \\ \Im\{x_j[S(n+1)-1]\} \end{bmatrix}}_{\bar{\mathbf{x}}_j[n]} + \underbrace{\begin{bmatrix} \Re\{r_i[Sn]\} \\ \Im\{r_i[Sn]\} \\ \vdots \\ \Re\{r_i[S(n+1)-1]\} \\ \Im\{r_i[S(n+1)-1]\} \end{bmatrix}}_{\bar{\mathbf{r}}_i[n]} \tag{4}$$

where $\otimes$ denotes Kronecker product and $\mathbf{I}_S$ is the identity matrix of size S. Note that $\bar{\mathbf{R}}(\phi_{ij})$ also satisfies (3).

The ith transmitter applies a precoding matrix that we denote as $\mathbf{V}_i = [\mathbf{v}_i^1, \ldots, \mathbf{v}_i^{d_i}] \in \mathbb{R}^{2S \times d_i}$, where d_i is the number of real symbols transmitted by user i. Therefore, the signal transmitted by user i is given by

$$\bar{\mathbf{x}}_i = \mathbf{V}_i \mathbf{s}_i \ \in \mathbb{R}^{2S} \ , \tag{5}$$

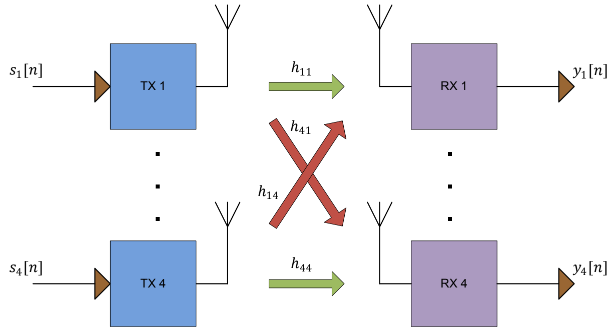

Fig. 1. The 4-user SISO interference channel.

where we have dropped the time index for the sake of clarity. In (5), $\mathbf{s}_i \in \mathbb{R}^{d_i}$ is the vector of real transmitted symbols, which is distributed as $\mathcal{N}(\mathbf{0}, \mathbf{I}_{d_i})$.

It is interesting at this point to remind that if $\mathbf{V}_i$ had complex structure

$$\mathbf{V}_i = \begin{bmatrix} \Re\{\mathbf{A}\} & -\Im\{\mathbf{A}\} \\ \Im\{\mathbf{A}\} & \Re\{\mathbf{A}\} \end{bmatrix}, \tag{6}$$

with $\mathbf{A}$ being a matrix of the appropriate dimensions, then the transmitted signal $\bar{\mathbf{x}}_i$ would be proper and, consequently, its pseudo-covariance matrix would vanish to zero, i.e., $\mathbb{E}[\bar{\mathcal{X}}_i \bar{\mathcal{X}}_i^T] = 0$ [8], where $\bar{\mathcal{X}}_i$ is the complex form of $\bar{\mathbf{x}}_i$, i.e., $\bar{\mathcal{X}}_i = [x_i[n], \ldots, x_i[n-S+1]]^T$. In model (5), however, no particular structure is imposed on the precoding matrices and therefore the transmitted signals are improper. This also provides more freedom in the design of the IA precoders. At the receiver side, we apply a linear decoder (again without any particular structure) to suppress the interference, $\mathbf{U}_i \in \mathbb{R}^{2S \times d_i}$. Hence, the output signal at the ith receiver can be expressed as

$$\begin{aligned} \mathbf{z}_i = &\mathbf{U}_i^H |h_{ii}| \bar{\mathbf{R}}(\phi_{ii}) \mathbf{V}_i \mathbf{s}_i + \mathbf{U}_i^H \sum_{j \neq i} |h_{ij}| \bar{\mathbf{R}}(\phi_{ij}) \mathbf{V}_j \mathbf{s}_j \\ &+ \mathbf{U}_i^H \bar{\mathbf{r}}_i \ . \end{aligned} \tag{7}$$

With this setting, interference alignment is achieved if the following conditions hold

$$\left\| \mathbf{U}_i^H \bar{\mathbf{R}}(\phi_{ij}) \mathbf{V}_j \right\| = 0 \ , \ i, j \in \{1, \ldots, 4 | i \neq j\} \ , \tag{8}$$

$$\operatorname{rank}\left(\mathbf{U}_i^H \bar{\mathbf{R}}(\phi_{ii}) \mathbf{V}_i \right) = d_i \ , \ i = 1, \ldots, 4 \tag{9}$$

and then the total number of DoF is given by

$$\text{DoF} = \frac{d_1 + d_2 + d_3 + d_4}{2S}. \tag{10}$$

III. Preliminary Results

In [3], a fundamental limitation of the 3-user channel with improper signaling, stating that a signal vector cannot align at more than one undesired receiver without becoming aligned at its own desired receiver, was shown to exist. This limitation is in the end responsible for the impossibility of surpassing the 1.2 DoF for this scenario[2]. and is a consequence of the

[2]Actually, the 3-user channel has 1.5 DoF for a subset of channel coefficients of measure zero [3]. For generic channels, however, we cannot achieve more than 1.2 DoF.

following lemma, taken from [3] and reproduced here for completeness.

Lemma 1: For any given complex vector $\mathbf{v}$, and for any given angles, α, β, such that

$$\sin(\alpha-\beta) \neq 0 , \tag{11}$$

there exist real constants $\{g_1, g_2\} \in \mathbb{R}$ such that

$$\mathbf{v} = g_1 \mathbf{v} e^{\jmath\alpha} + g_2 \mathbf{v} e^{\jmath\beta} . \tag{12}$$

Proof: See [3]. ∎

In the following lemma, we prove that this limitation does not exist for the 4-user interference channel with constant coefficients, thus opening the possibility of achieving more than 1.2 DoF for this channel.

Lemma 2: Let us assume that $d_1 = d_2 = d_3 = d_4 = d \geq 2$ and suppose that $\mathbf{v}_1^1$ is aligned within the interference subspace at all undesired receivers, i.e.,

$$\text{At Rx 2: } \bar{\mathbf{R}}(\phi_{21})\mathbf{v}_1^1 \in \text{span}\left(\left[\bar{\mathbf{R}}(\phi_{23})\mathbf{V}_3\ ,\ \bar{\mathbf{R}}(\phi_{24})\mathbf{V}_4\right]\right) , \tag{13}$$

$$\text{At Rx 3: } \bar{\mathbf{R}}(\phi_{31})\mathbf{v}_1^1 \in \text{span}\left(\left[\bar{\mathbf{R}}(\phi_{32})\mathbf{V}_2\ ,\ \bar{\mathbf{R}}(\phi_{34})\mathbf{V}_4\right]\right) , \tag{14}$$

$$\text{At Rx 4: } \bar{\mathbf{R}}(\phi_{41})\mathbf{v}_1^1 \in \text{span}\left(\left[\bar{\mathbf{R}}(\phi_{42})\mathbf{V}_2\ ,\ \bar{\mathbf{R}}(\phi_{43})\mathbf{V}_3\right]\right) . \tag{15}$$

Let us also suppose that transmitter 2 aligns all its data streams with interference at receiver 1, i.e.,

$$\text{At Rx 1: } \bar{\mathbf{R}}(\phi_{12})\mathbf{V}_2 \in \text{span}\left(\left[\bar{\mathbf{R}}(\phi_{13})\mathbf{V}_3\ ,\ \bar{\mathbf{R}}(\phi_{14})\mathbf{V}_4\right]\right) , \tag{16}$$

Then, except for a subset of channel coefficients with measure zero, $\mathbf{v}_1^1$ can be distinguishable at the desired receiver, i.e., linearly independent of the interference subspace

$$\bar{\mathbf{R}}(\phi_{11})\mathbf{v}_1^1 \notin \text{span}\left(\left[\bar{\mathbf{R}}(\phi_{13})\mathbf{V}_3\ ,\ \bar{\mathbf{R}}(\phi_{14})\mathbf{V}_4\right]\right) . \tag{17}$$

Proof: (13)–(16) can be expressed as

$$e^{\jmath\phi_{21}}\mathcal{V}_1^1 = \mathcal{V}_3 e^{\jmath\phi_{23}}\mathbf{a} + \mathcal{V}_4 e^{\jmath\phi_{24}}\mathbf{b} , \tag{18}$$

$$e^{\jmath\phi_{31}}\mathcal{V}_1^1 = \mathcal{V}_2 e^{\jmath\phi_{32}}\mathbf{c} + \mathcal{V}_4 e^{\jmath\phi_{34}}\mathbf{f} , \tag{19}$$

$$e^{\jmath\phi_{41}}\mathcal{V}_1^1 = \mathcal{V}_2 e^{\jmath\phi_{42}}\mathbf{k} + \mathcal{V}_3 e^{\jmath\phi_{43}}\mathbf{l} , \tag{20}$$

$$e^{\jmath\phi_{12}}\mathcal{V}_2 = \mathcal{V}_3 e^{\jmath\phi_{13}}\mathbf{M} + \mathcal{V}_4 e^{\jmath\phi_{14}}\mathbf{N} , \tag{21}$$

where $\mathcal{V}_j$ ($\mathcal{V}_j^1$) is the complex form of $\mathbf{V}_j$ ($\mathbf{v}_j^1$), $\{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{f}, \mathbf{k}, \mathbf{l}\} \in \mathbb{R}^{d\times 1}$ and $\{\mathbf{M}, \mathbf{N}\} \in \mathbb{R}^{d\times d}$. Now consider the following equality

$$e^{\jmath\phi_{11}}\mathcal{V}_1^1 = g_1\mathcal{V}_1^1 + g_2\mathcal{V}_1 + g_3\mathcal{V}_1^1 , \tag{22}$$

which holds for any given $\{g_1, g_2, g_3\}$ such that $g_1+g_2+g_3 = e^{\jmath\phi_{11}}$. Then, substituting (18)–(20) into (22), and considering all possible vectors within the signal subspace yields

$$\begin{aligned} e^{\jmath\phi_{11}}\mathcal{V}_1^1 =& e^{\jmath\phi_{13}}\mathcal{V}_3\Big[g_1 e^{\jmath(\phi_{23}-\phi_{21}-\phi_{13})}\mathbf{a} + g_3\mathbf{l}' + \\ & g_2 e^{\jmath(\phi_{32}-\phi_{31}-\phi_{12})}\mathbf{Mc}\Big] + e^{\jmath\phi_{14}}\mathcal{V}_4\Big[g_2\mathbf{f}' + \\ & g_1 e^{\jmath(\phi_{24}-\phi_{21}-\phi_{14})}\mathbf{b} + g_3 e^{\jmath(\phi_{42}-\phi_{41}-\phi_{12})}\mathbf{Nk}\Big] . \end{aligned} \tag{23}$$

where $\mathbf{l}' = e^{\jmath(\phi_{43}-\phi_{41}-\phi_{13})}\mathbf{l} + e^{\jmath(\phi_{42}-\phi_{41}-\phi_{12})}\mathbf{Mk}$ and $\mathbf{f}' = e^{\jmath(\phi_{34}-\phi_{31}-\phi_{14})}\mathbf{f} + e^{\jmath(\phi_{32}-\phi_{31}-\phi_{12})}\mathbf{Nc}$. In order for the right-hand side of (23) to be in the interference subspace at receiver 1, the following must hold

$$\Im\left\{g_1 e^{\jmath(\phi_{23}-\phi_{21}-\phi_{13})}\mathbf{a} + g_2 e^{\jmath(\phi_{32}-\phi_{31}-\phi_{12})}\mathbf{Mc} + g_3\mathbf{l}'\right\} = \mathbf{0} , \tag{24}$$

$$\Im\left\{g_1 e^{\jmath(\phi_{24}-\phi_{21}-\phi_{14})}\mathbf{b} + g_3 e^{\jmath(\phi_{42}-\phi_{41}-\phi_{12})}\mathbf{Nk} + g_2\mathbf{f}'\right\} = \mathbf{0} . \tag{25}$$

Notice that, for general values of $\{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{f}, \mathbf{k}, \mathbf{l}, \mathbf{M}, \mathbf{N}\}$, there are values of $\{g_1, g_2, g_3\}$ satisfying (22), (24) and (25). However, except for a subset of channel coefficients of measure zero, the aforementioned coefficients can be selected such that the linear system becomes inconsistent, thus making it possible to distinguish the desired signal from interference and, consequently, proving the lemma. To this end, let us consider the following linear combination of (24) and (25)

$$\begin{aligned} \Im\Big\{& g_1\left(e^{\jmath(\phi_{23}-\phi_{21}-\phi_{13})}\mathbf{1}^T\mathbf{a} + e^{\jmath(\phi_{24}-\phi_{21}-\phi_{14})}\mathbf{1}^T\mathbf{b}\right) + \\ & g_2\left(e^{\jmath(\phi_{32}-\phi_{31}-\phi_{12})}\mathbf{1}^T\mathbf{Mc} + \mathbf{1}^T\mathbf{f}'\right) + \\ & g_3\left(\mathbf{1}^T\mathbf{l}' + e^{\jmath(\phi_{42}-\phi_{41}-\phi_{12})}\mathbf{1}^T\mathbf{Nk}\right)\Big\} = 0 . \end{aligned} \tag{26}$$

Consider now the coefficients of g_1 in (26). According to Lemma 1, there is a value of $\mathbf{1}^T\mathbf{a}$ and $\mathbf{1}^T\mathbf{b}$ such that $e^{\jmath(\phi_{23}-\phi_{21}-\phi_{13})}\mathbf{1}^T\mathbf{a} + e^{\jmath(\phi_{24}-\phi_{21}-\phi_{14})}\mathbf{1}^T\mathbf{b} = 1$, unless $\sin(\phi_{23}-\phi_{13}+\phi_{14}) = 0$, which only holds for a subset of channel coefficients that has zero measure. Applying the same idea to the coefficients of g_2 and g_3, (26) is reduced to

$$\Im\{g_1 + g_2 + g_3\} = 0 . \tag{27}$$

As there are no values of $\{g_1, g_2, g_3\}$ satisfying both (22) and (27), the lemma is proved. ∎

Lemma 2 states that any given signal vector can actually align with the interference at more than one undesired receiver without becoming aligned within the interference subspace at its own desired receivers, thus breaking the limitation established in [3].

From the proof of Lemma 2, another difference between the 3-user and the 4-user channels becomes evident. While for the 3-user channel IA is achieved by individual or one-to-one alignments (by this we mean that two interfering vectors coming from two different users align along the same direction), for the 4-user channel each interfering vector must be a linear combination of *all* basis vectors of the interference subspace. This fact was also observed in [6] for the 4-user single-input multiple-output (SIMO) channel with time-varying coefficients.

IV. DoF Inner and Outer Bounds

In this section we present the main results of this work, which are formalized in the following theorem.

Theorem 1: Using the class of linear interference alignment schemes described in this paper, the maximum number of DoF that are achievable in the 4-user complex Gaussian interference channel with constant channel coefficients, is bounded by

$$\frac{4}{3} \leq \frac{d_1 + d_2 + d_3 + d_4}{2S} \leq \frac{8}{5} \; . \qquad (28)$$

We prove the theorem in the following subsections.

A. Converse for Theorem 1

Proof: Let us take any 3 users among the 4 users to form a 3-user SISO interference channel. According to [3], asymmetric complex signaling is able to achieve a maximum of $\frac{6}{5}$ DoF, therefore

$$\frac{d_i + d_j + d_k}{2S} \leq \frac{6}{5} \quad \forall i, j, k \in \{1, 2, 3, 4\} \; , \; i \neq j \neq k \quad (29)$$

hold. Adding up the above expressions yields

$$3\frac{d_1 + d_2 + d_3 + d_4}{2S} \leq \frac{24}{5} \Rightarrow \frac{d_1 + d_2 + d_3 + d_4}{2S} \leq \frac{8}{5} \; , \qquad (30)$$

hence proving the outer bound. ■

B. Achievability Proof for Theorem 1

Proof: Let us suppose that $\mathbf{v}_1^1$ aligns at all undesired receivers. Therefore, (13)–(15) hold. Consequently, $\mathbf{v}_1^1$ must lie in the intersection of 3 subspaces, i.e.,

$$\begin{aligned} \mathbf{v}_1^1 \in \text{span}\left(\left[\bar{\mathbf{R}}\left(\phi_{23} - \phi_{21}\right)\mathbf{V}_3 \; , \; \bar{\mathbf{R}}\left(\phi_{24} - \phi_{21}\right)\mathbf{V}_4\right]\right) \cap \\ \text{span}\left(\left[\bar{\mathbf{R}}\left(\phi_{32} - \phi_{31}\right)\mathbf{V}_2 \; , \; \bar{\mathbf{R}}\left(\phi_{34} - \phi_{31}\right)\mathbf{V}_4\right]\right) \cap \\ \text{span}\left(\left[\bar{\mathbf{R}}\left(\phi_{42} - \phi_{41}\right)\mathbf{V}_2 \; , \; \bar{\mathbf{R}}\left(\phi_{43} - \phi_{41}\right)\mathbf{V}_3\right]\right) \; . \end{aligned} \qquad (31)$$

Hence, the above subspaces must have an intersection of at least one dimension. More generally, the number of streams that user 1 is able to align simultaneously at the 3 undesired receivers is equal to the dimension of the intersection (31). This allow us to interpret the alignment for the 4-user channel in a particular way: there are users who spend their DoF in creating intersections, whereas other users spend their DoF making their streams to lie within these intersections.

Now the question is how many DoF must be spent to create an intersection. To answer this question, let us consider three arbitrary subspaces, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, such that $\mathcal{A} \cap \mathcal{B} \cap \mathcal{C} = \emptyset$. In this situation, a maximum of two basis vectors of two different subspaces, e.g. $\mathbf{a} \in \mathcal{A}$ and $\mathbf{b} \in \mathcal{B}$, must be modified so as to create an intersection of one dimension: $\dim(\mathcal{A} \cap \mathcal{B} \cap \mathcal{C}) = 1$. Taking this into account, we consider now a transmission scheme where each user wishes to send d real streams over S symbol extensions, hence yielding $\frac{4d}{2S}$ DoF. In this situation, users 3 and 4 design their precoders (spending each one d dimensions) to create an intersection of dimension d for user 1, who can now design its precoder to fall within this intersection thus aligning its d streams at all undesired receivers. Finally, user 2 can still align its d streams within the interference subspace at receiver 1, while it is also aligned at receivers 3 and 4 because it participates in the intersection (31). Notice that, according to Lemma 2, the desired signal can be distinguishable from interference at the desired receivers. With this setting, the interference subspace at any receiver is comprised of $3d$ vectors that overlap in d dimensions, hence creating a $2d$-dimensional subspace. Since the vectors belong to $\mathbb{R}^{2S}$, for this partitioning to be feasible while still leaving d dimensions at each receiver for the desired signal, the sum of dimensions of the interference and desired signal subspaces must not exceed the total dimension of the signal space, i.e.,

$$3d \leq 2S \Rightarrow \frac{4d}{2S} \leq \frac{4}{3} \; . \qquad (32)$$

Since this scheme is able to achieve $\frac{4}{3}$ DoF, the inner bound is therefore proved. ■

C. Remarks

1) It is worth pointing out that, unlike other schemes such as the one presented in [6] for the SIMO case, which requires $3(n+1)^8$ symbol extensions to achieve $\frac{8}{3}$ DoF, $n \in \mathbb{N}$; our proposed scheme is able to achieve the inner bound of $\frac{4}{3}$ DoF by using only $S = 3$ symbol extensions. Furthermore, it also requires less symbol extensions than the 3-user case, where $S = 5$ is needed to get 1.2 DoF [3].

2) It is important to notice that, to obtain the achievability of $\frac{4}{3}$ DoF, the interference from each transmitter aligns, in general, within the subspace spanned by the rest of interference vectors, i.e., every entry of $\{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{f}, \mathbf{k}, \mathbf{l}, \mathbf{M}, \mathbf{N}\}$ is not equal to zero in (18)–(21). To increase the achievable DoF beyond $\frac{4}{3}$, the interference must be confined into a subspace with fewer dimensions, thus allowing more signals to be transmitted. Nevertheless, we conjecture that further reducing the dimension of the interference subspace makes the desired signals undistinguishable from the interference. In this case, $\frac{4}{3}$ would be the maximum number of DoF that are achievable using linear schemes. A formal proof, however, is yet to be found.

V. Numerical Examples

In this section we first present some numerical examples to show the performance improvement of IA with improper signaling scheme over conventional TDMA. We consider a 4-user, fully connected, Gaussian interference channel with constant channel coefficients, where each user sends 1 complex stream of data in $S = 3$ symbol extensions, hence $\frac{4}{3}$ DoF are achieved. To obtain the IA precoders and decoders, we resort to an alternating minimization algorithm similar to that used in the proper signaling case [9], [10]. Basically, the algorithm is the same but using only real quantities, with the channels having the structure in (1), while the precoders and decoders are unstructured, thus yielding improper signals. Another difference is that, in the proper case, choosing unitary precoders and decoders guarantees (9) with probability one. In the improper case, however, this is not true anymore because the structured real channels in (1) are not generic. To solve this minor issue the alternating minimization algorithm has to be initialized at different points until a feasible solution satisfying both (8) and (9) is achieved.

In Fig. 2 we compare the sum-rate performance of TDMA with proper signaling and IA with improper signaling. The

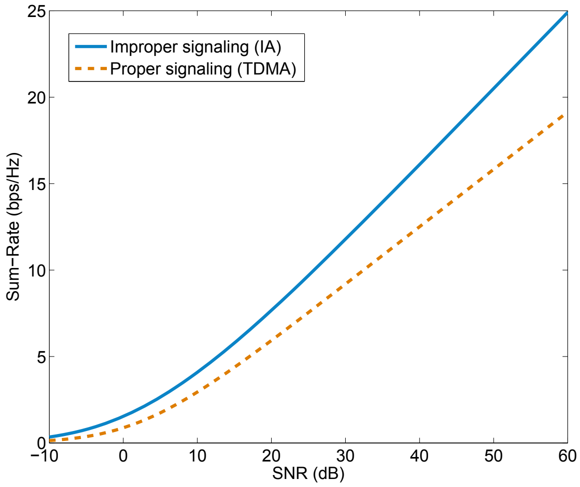

Fig. 2. Sum-rate performance using TDMA (proper signaling) and IA with improper signaling. Results averaged over 1000 random channel realizations.

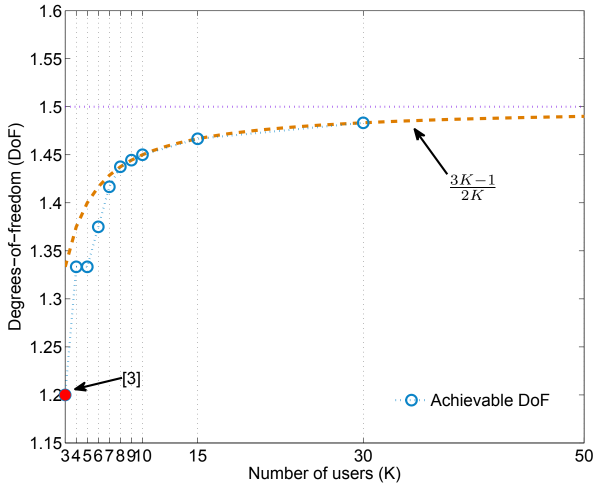

Fig. 3. Achievable DoF obtained numerically through an alternating minimization algorithm.

improvement is evident not only at high SNRs, but even in the medium and low SNR regimes. Finally, we have applied the aforementioned alternating minimization algorithm to study the achievable DoF when $K > 4$. Although this is by no means an achievability proof, the results in Fig. 3 provide us with an intuition on how the DoF behave with respect to the number of users for Gaussian interference channel with constant coefficients when improper signaling is used. We observe that the achievable DoF increases with the number of users, and seem to tend asymptotically to 1.5. Furthermore, the obtained results match perfectly the curve $\frac{3K-1}{2K}$ when $K \geq 8$. Note that we have not been able to obtain more than $\frac{4}{3}$ DoF for the 4-user channel, which supports our conjecture that the derived inner bound is tight.

VI. Conclusions and Future Work

In this work, we have shown that using linear interference alignment and improper signaling at least $\frac{4}{3}$ DoF and no more than $\frac{8}{5}$ DoF are achievable for the 4-user SISO interference channel with constant channel coefficients. We have shown that the limitation of improper signaling for the 3-user channel that was pointed out in [3], does not exist for the 4-user channel. In consequence, the 1.2 DoF limit established for the 3-user channel can be surpassed when 4 users share the channel. We conjecture that the inner bound of $\frac{4}{3}$ DoF is in fact tight. To rigorously proof this conjecture is our future work.

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