A Fast Algorithm for Designing Grassmannian Constellations

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***Abstract*—In this paper, we propose an algorithm for designing Grassmannian constellations for noncoherent MIMO communications over Rayleigh block fading channels. The new algorithm, named ManOpt, is based on a gradient ascent approach that operates directly on the Grassmann manifold to maximize the minimum pairwise chordal distance of the constellation. We analyze the performance of the algorithm in terms of convergence speed, minimum chordal distance achieved, and symbol error rate (SER), and we compare it with some other Grassmannian constellation designs. Our simulation results suggest that ManOpt constellations have a higher packing efficiency (represented by the minimum chordal distance), which translates into better SER performance, with lower computational complexity than existing algorithms for unstructured constellation designs.**

***Index Terms*—Noncoherent, MIMO communications, Grassmannian constellations.**

I. Introduction

In multiple-input multiple-output (MIMO) communications systems, the channel state information (CSI) is typically estimated at the receiver side by sending a few known pilots and then used for decoding at the receiver and/or for precoding at the transmitter. These are known as coherent schemes and in slowly fading scenarios, when the channel remains approximately constant over a long coherence time, the MIMO channel capacity for coherent systems is known to increase linearly with the minimum number of transmit and receive antennas at high signal-to-noise (SNR) ratio [1], [2]. In fast fading scenarios, however, to obtain an accurate channel estimate would require pilots to occupy a disproportionate fraction of communication resources. Even in slowly-varying channels, CSI acquisition by orthogonal pilot-based schemes can result in significant overheads in massive MIMO systems [3], and the performance of coherent massive MIMO systems can be degraded by channel aging [4]. These scenarios motivate the use of noncoherent MIMO communications schemes in which neither the transmitter nor the receiver have any knowledge about the instantaneous CSI (although they might have some knowledge about the statistical or long-term CSI such as its fading distribution).

Despite the absence of *instantaneous* CSI at the receiver, noncoherent MIMO communication systems can achieve a significant fraction of the coherent capacity at high SNR, as shown in [5]–[7]. Under additive Gaussian noise and assuming a Rayleigh block-fading channel, these works proved that at high SNR and when the coherence interval, T, is larger than or equal to twice the number of transmit antennas M ($T \geq 2M$), capacity can be achieved by transmitting isotropically distributed unitary matrices. The pre-log factor in the high-SNR capacity expression is $M(1-M/T)$, so the noncoherent multiplexing gain approaches the coherent multiplexing gain as $T \to \infty$. Codebooks formed by isotropically distributed unitary space-time matrices correspond to optimal packings in Grassmann manifolds [7], [8]. Therefore, in noncoherent MIMO communication systems the information is carried by the column span of the transmitted $T \times M$ matrix, $\mathbf{X}$, which is not affected by the MIMO channel $\mathbf{H}$. In other words, the column span of $\mathbf{X}$ is identical to the column space of $\mathbf{XH}$.

There has been extensive research on the design of noncoherent structured and unstructured constellations as optimal packings on the Grassmann manifold [9]–[14]. Some experimental evaluation of Grassmannian constellations in noncoherent communications using over-the-air transmission has been reported in [15]. Existing constellation designs can be generically categorized into two groups: structured or unstructured. Structured designs impose some kind of structure on the constellation points through parameterized mappings such as the exp-map design in [11], algebraic constructions such as the Fourier-based constellation in [16], structured partitions of the Grassmannian like the recently proposed cube-split constellation [12], or designs based on group representations [17], [18]. Structured designs may simplify either codeword generation/storage or detection, but the packing efficiency is lower than that achieved by unstructured codes, which in turn translates into poorer performance in terms of symbol error rate (SER). For this reason, in this paper we consider unstructured designs based on numerical optimization methods.

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Among the unstructured designs we can mention the alternating projection method [13], which enforces in each iteration both structural and spectral properties of the Gram matrix formed by the inner products between codewords, as well as the numerical methods in [9], [10], which minimize certain distance measures on the Grassmannian. For example, [9] employs as a suitable distance metric the chordal distance between subspaces, while [10] uses the spectral distance (the cosine square of the minimum principal angle). The method proposed in [9] is computationally costly because it works on a Grassmann manifold whose dimension grows with the number of constellation points. In this paper, we directly maximize the minimum pairwise chordal distance by means of a gradient ascent algorithm on the manifold that incorporates an adaptive adaptation step. Overall, the proposed method is computationally more efficient compared to other unstructured designs, and achieves packings that approximate the existing upper bounds and, in some cases, outperforms the best packings known to date [19].

In the following we provide a detailed description of the data model, the optimization problem and the proposed algorithm, as well as some simulation results to assess its convergence speed, the minimum chordal distance achieved and the symbol error rate (SER) performance compared with other Grassmannian packings proposed in the literature.

II. System Model

We consider a noncoherent MIMO communication system in which a transmitter with M antennas transmits to a receiver equipped with N antennas over a frequency-flat block-fading channel with coherence time T symbol periods, with $T \geq 2M$. That is, the channel matrix $\mathbf{H} \in \mathbb{C}^{M\times N}$ remains constant during each coherence block of T symbols, and changes to an independent realization in the next block. The MIMO channel $\mathbf{H}$ is assumed to be Rayleigh with entries $h_{ij} \sim \mathcal{CN}(0,1)$ and unknown to both the transmitter and the receiver.

Within a coherence block, the transmitter sends a unitary matrix $\mathbf{X} \in \mathbb{C}^{T\times M}$, $\mathbf{X}^H\mathbf{X} = \mathbf{I}_M$, that is a basis for the subspace $\langle \mathbf{X} \rangle \in \mathbb{G}\left(M, \mathbb{C}^T\right)$, where $\mathbb{G}\left(M, \mathbb{C}^T\right)$ denotes the Grassmann manifold of M-dimensional subspaces on $\mathbb{C}^T$. The signal at the receiver $\mathbf{Y} \in \mathbb{C}^{T\times N}$ is

$$\mathbf{Y} = \mathbf{X}\mathbf{H} + \sqrt{\frac{M}{T\rho}}\mathbf{W}, \tag{1}$$

where $\mathbf{W} \in \mathbb{C}^{T\times N}$ represents the additive Gaussian noise, modeled as $w_{ij} \sim \mathcal{CN}(0,1)$, and ρ represents the signal-to-noise-ratio (SNR).

The optimal detector is the maximum likelihood (ML) detector, given by

$$\tilde{\mathbf{X}} = \arg\max_{\mathbf{X}\in\mathcal{C}} \operatorname{tr}\left(\mathbf{Y}^H\mathbf{P}_{\mathbf{X}}\mathbf{Y}\right), \tag{2}$$

where $\operatorname{tr}(\mathbf{X})$ denotes trace of $\mathbf{X}$, $\mathcal{C}$ represents the codebook of K codewords and $\mathbf{P}_{\mathbf{X}} = \mathbf{X}\mathbf{X}^H$ is the projection matrix to the subspace $\langle \mathbf{X} \rangle$. Each codeword carries $\log_2(K)$ bits of information.

III. Proposed codebook design

A. Preliminaries

Natural geometric objects of relevance to this problem are the Stiefel and Grassmann manifolds. The Stiefel manifold $\mathbb{S}_t(M, \mathbb{C}^T)$ is defined as the set of all M-dimensional frames (semi-unitary $T \times M$ matrices) in $\mathbb{C}^T$

$$\mathbb{S}_t(M, \mathbb{C}^T) = \{\mathbf{X} \in \mathbb{C}^{T\times M} : \mathbf{X}\mathbf{X}^H = \mathbf{I}_M\}. \tag{3}$$

The Grassmann manifold $\mathbb{G}(M, \mathbb{C}^T)$ is the set of M-dimensional subspaces on $\mathbb{C}^T$, that is a complex manifold of dimension $M(T-M)$ [20]. Elements of $\mathbb{G}(M, \mathbb{C}^T)$ are represented by matrices in the Stiefel manifold $\mathbf{X} \in \mathbb{S}_t(M, \mathbb{C}^T)$. This representation is not unique since $\mathbf{X}$ and $\mathbf{X}\mathbf{U}$, with $\mathbf{U} \in \mathcal{U}(M)$ a unitary $M \times M$ matrix, represent the same element in $\mathbb{G}(M, \mathbb{C}^T)$, so formally we should denote elements of the Grassmannian as $\langle \mathbf{X} \rangle$ where $\mathbf{X} \in \mathbb{S}_t(M, \mathbb{C}^T)$ and $\langle \mathbf{X} \rangle$ is the class of equivalence of $\mathbf{X}$ under the quotient by the set of $\mathcal{U}(M)$.

There are different ways of measuring the distance between subspaces, all of them being ultimately different functions of the principal angles between subspaces. Suppose that $\mathbf{X}_k$ and $\mathbf{X}_j$ are two matrices whose columns form orthonormal bases for the subspaces $\langle \mathbf{X}_k \rangle$ and $\langle \mathbf{X}_j \rangle$, which are elements in $\mathbb{G}(M, \mathbb{C}^T)$. Then, the singular values of $\mathbf{X}_k^H\mathbf{X}_j$ are $(\cos\theta_1, \ldots, \cos\theta_M)$, where $\theta_1, \ldots, \theta_M$ are the principal angles [21].

In the context of non-coherent communications, the chordal distance (a.k.a. projection F-norm distance [22]) is of particular relevance, mainly because of its mathematical tractability. It is defined as

$$\begin{aligned} d\left(\langle \mathbf{X}_k \rangle, \langle \mathbf{X}_j \rangle\right) &= \frac{1}{\sqrt{2}}\|\mathbf{X}_k\mathbf{X}_k^H - \mathbf{X}_j\mathbf{X}_j^H\|_F \\ &= \sqrt{\sum_{i=1}^{M}\sin^2\theta_i}, \end{aligned} \tag{4}$$

where $\|\mathbf{A}\|_F$ denotes the Frobenius norm of $\mathbf{A}$. This will be the distance measure used in this work.

B. Cost function

It was shown in [5]–[7] that the high-SNR capacity for noncoherent MIMO systems is achieved by transmitting isotropically distributed unitary space-time matrices $\mathbf{X}$, that is, transmitting subspaces that are uniformly distributed on the Grassmann manifold $\mathbb{G}(M, \mathbb{C}^T)$.

Motivated by this fact, the constellation $\mathcal{C}$ can be designed by choosing $|\mathcal{C}| = K$ elements of $\mathbb{G}(M, \mathbb{C}^T)$, represented by their orthonormal bases $\{\mathbf{X}_1, \ldots, \mathbf{X}_K\}$ with certain desirable distance properties.

Note that both $\mathbf{X}$ and the noise-free observation $\mathbf{X}\mathbf{H}$ represent the same symbol in $\mathbb{G}(M, \mathbb{C}^T)$. Therefore, Grassmannian signaling guarantees error-free detection without CSI in the noiseless case. When the noise $\mathbf{W}$ is present, since its columns are almost surely not aligned with the transmitted signal $\mathbf{X}$, the span of the received signal $\mathbf{Y}$ deviates from that of $\mathbf{X}$ with respect to a distance measure, leading to a detection error if $\mathbf{Y}$ is outside the decision region of the transmitted symbol.

Consequently, our main goal is to design a codebook $\mathcal{C} = \{\mathbf{X}_1, \ldots, \mathbf{X}_K\}$ that minimizes the symbol error rate (SER). As usually, we use a pairwise error probability bound instead of the SER for simplicity [23]. In this case, we are limited by the worst pairwise error probability, which depends on the minimum pairwise distance between codewords or subspaces. Although the dependence of the pairwise error on the distance between subspaces is in general difficult to establish, different approximations have appeared in the literature [16], [23]. The authors in [16] provided an upper bound on the pairwise probability of error in terms of the chordal distance between subspaces. Since then, the chordal distance has been the most commonly employed measure for the design of unstructured Grassmannian constellations. Following these works, we consider the following optimization problem

$$\begin{aligned} \max_{\mathbf{X}_1, \ldots, \mathbf{X}_K} \quad & \min_{k \neq j} d\left(\langle \mathbf{X}_k \rangle, \langle \mathbf{X}_j \rangle\right) \\ \text{s.t.} \quad & \mathbf{X}_k^H \mathbf{X}_k = \mathbf{I}_M, \\ & \mathbf{X}_k \in \mathbb{C}^{T \times M}, \quad k = 1, \ldots, K \end{aligned} \tag{5}$$

where $d\left(\langle \mathbf{X}_k \rangle, \langle \mathbf{X}_j \rangle\right)$ is the chordal distance defined in (4). A closely related cost function has been considered in [9]. The optimization carried out in [9] replaces the $\max$ function by a smooth surrogate that couples all pairwise chordal distances and then applies an optimization algorithm on a Grassmann manifold of dimension $\mathbb{G}(M|\mathcal{C}|, \mathbb{C}^{T|\mathcal{C}|})$, where $|\mathcal{C}|$ denotes the cardinality of the constellation. Thus, the computational complexity of this method grows rapidly with the number of codewords. In the following subsection, we describe a simple optimization technique at a much lower computational cost.

C. Manifold optimization (ManOpt)

To solve (5), we propose a gradient ascent approach that operates directly on the Grassmann manifold $\mathbb{G}(M, \mathbb{C}^T)$. The algorithm starts from a random collection of K elements of the Grassmann manifold $\mathbb{G}(M, \mathbb{C}^T)$ and, at each iteration, tries to separate as much as possible the closest codewords according to the chordal distance. The optimization is performed on the manifold, so the essential technical aspect of the algorithm is the calculation of the gradient in the tangent space, which we detail below.

As we have already discussed, points on $\mathbb{G}(M, \mathbb{C}^T)$ are equivalence classes of $T \times M$ matrices, where the orthogonal bases for the subspaces $\langle \mathbf{X}_j \rangle$ and $\langle \mathbf{X}_k \rangle$ are equivalent if $\mathbf{X}_j = \mathbf{X}_k \mathbf{U}$ for some $\mathbf{U} \in \mathcal{U}(M)$. Therefore, the Grassmann manifold may be defined as a quotient space $\mathbb{S}_t(M, \mathbb{C}^T)/\mathcal{U}(M)$, where $\mathcal{U}(M)$ denotes the orthogonal group of dimension M. Mathematically, this defines a Riemannian structure in the Grassmannian given by the *Riemannian submersion*

$$\begin{array}{rccc} \varphi: & \mathbb{S}_t(M, \mathbb{C}^T) & \to & \mathbb{G}(M, \mathbb{C}^T) = \mathbb{S}_t(M, \mathbb{C}^T)/\mathcal{U}(M) \\ & \mathbf{X}_k & \mapsto & \langle \mathbf{X}_k \rangle. \end{array} \tag{6}$$

The tangent space to the Stiefel manifold at $\mathbf{X}_k \in \mathbb{S}_t(M, \mathbb{C}^T)$ is easy to describe from the defining equation $\mathbf{X}_k^H \mathbf{X}_k = \mathbf{I}_M$ given in (3)

$$\begin{aligned} T_{\mathbf{X}_k} \mathbb{S}_t(M, \mathbb{C}^T) = & \Big\{ \dot{\mathbf{X}}_k \in \mathbb{C}^{T \times M} : \frac{d}{dt} |_{t=0} \; ((\mathbf{X}_k + t\dot{\mathbf{X}}_k)^H \\ & \cdot (\mathbf{X}_k + t\dot{\mathbf{X}}_k)) = 0 \Big\} \\ = & \{ \dot{\mathbf{X}}_k \in \mathbb{C}^{T \times M} : \dot{\mathbf{X}}_k^H \mathbf{X}_k + \mathbf{X}_k^H \dot{\mathbf{X}}_k = 0 \}, \end{aligned} \tag{7}$$

where $\dot{\mathbf{X}}_k$ denotes an arbitrary tangent vector at the point $\mathbf{X}_k$. The Riemannian submersion φ allows us to identify the tangent space to the Grassmannian with the orthogonal complement to the kernel of the tangent map of φ

$$\begin{aligned} T_{\mathbf{X}_k} \mathbb{G}(M, \mathbb{C}^T) \equiv & \{ \dot{\mathbf{X}}_k \in T_{\mathbf{X}_k} \mathbb{S}_t(M, \mathbb{C}^T) : \dot{\mathbf{X}}_k \perp \; \mathbf{X}_k \dot{\mathbf{U}} \\ & \text{for all } \dot{\mathbf{U}} \in T_{\mathbf{I}_M} \mathcal{U}(M) \} \\ = & \{ (\mathbf{I}_T - \mathbf{X}_k \mathbf{X}_k^H) \dot{\mathbf{X}} : \dot{\mathbf{X}} \in \mathbb{C}^{T \times M} \}, \end{aligned} \tag{8}$$

which obviously does not depend on the chosen representative for $\langle \mathbf{X}_k \rangle$. In other words, (8) shows that the gradient of any given cost function on the tangent space at $\langle \mathbf{X}_k \rangle$ is the unconstrained gradient, $\dot{\mathbf{X}}$, projected onto the orthogonal complement of the column space of $\mathbf{X}_k$.

Once we know the tangent space to the Grassmannian, we can define the best direction to take one of its elements far from another one, which will be the gradient of the function $f(\langle \mathbf{X}_k \rangle) = d(\langle \mathbf{X}_j \rangle, \langle \mathbf{X}_k \rangle)^2$ at the point $\langle \mathbf{X}_k \rangle$. The defining property of the gradient $\nabla f(\langle \mathbf{X}_k \rangle)$ says that for all $\dot{\mathbf{X}}_k = (\mathbf{I}_T - \mathbf{X}_k \mathbf{X}_k^H) \dot{\mathbf{X}}$ we must have

$$\begin{aligned} \Re(\langle \nabla f(\langle \mathbf{X}_k \rangle), \dot{\mathbf{X}}_k \rangle_F) = & \frac{d}{dt} |_{t=0} \; \| \mathbf{X}_j \mathbf{X}_j^H - \\ & (\mathbf{X}_k + t\dot{\mathbf{X}}_k)(\mathbf{X}_k + t\dot{\mathbf{X}}_k)^H \|^2 \\ = & - 4\Re(\langle (\mathbf{I}_T - \mathbf{X}_k \mathbf{X}_k^H) \mathbf{X}_j \mathbf{X}_j^H \mathbf{X}_k, \\ & \dot{\mathbf{X}}_k \rangle_F), \end{aligned} \tag{9}$$

where $\Re(\cdot)$ denotes real part, which leads us to

$$\nabla f(\langle \mathbf{X}_k \rangle) = -4(\mathbf{I}_T - \mathbf{X}_k \mathbf{X}_k^H) \mathbf{X}_j \mathbf{X}_j^H \mathbf{X}_k, \tag{10}$$

which is the gradient of $d(\langle \mathbf{X}_j \rangle, \langle \mathbf{X}_k \rangle)^2$ at $\langle \mathbf{X}_k \rangle$.

At each iteration, the algorithm performs the following steps:

1) Perform a random permutation of the codebook $\mathcal{C} = \{\mathbf{X}_1, \ldots, \mathbf{X}_K\}$, so that the optimization algorithm at each iteration changes the starting codeword. In this way, the method obtains better results in terms of convergence speed and minimum chordal distance.
2) For $k = 1, \ldots, K$, find the closest element $\mathbf{X}_j$ to the kth codeword, $\mathbf{X}_k$, and construct the matrix $\mathbf{\Delta}_{kj}$ that yields the best direction to get $\mathbf{X}_k$ far from $\mathbf{X}_j$ according to (10)

$$\mathbf{\Delta}_{kj} = -\frac{\mathbf{P}_{\mathbf{X}_k^\perp} \mathbf{P}_{\mathbf{X}_j} \mathbf{X}_k}{\| \mathbf{P}_{\mathbf{X}_k^\perp} \mathbf{P}_{\mathbf{X}_j} \mathbf{X}_k \|_F}, \tag{11}$$

where $\mathbf{P}_{\mathbf{X}_k^\perp} = \mathbf{I}_T - \mathbf{X}_k \mathbf{X}_k^H$ is the projection matrix onto the orthogonal complement of $\langle \mathbf{X}_k \rangle$ and $\mathbf{P}_{\mathbf{X}_j} = \mathbf{X}_j \mathbf{X}_j^H$ is the projection matrix onto $\langle \mathbf{X}_j \rangle$.

3) Move each $\mathbf{X}_k$ a certain amount, defined by the step-size μ, in the direction defined by the gradient

$$\tilde{\mathbf{X}}_k = \mathbf{X}_k + \mu \mathbf{\Delta}_{kj}. \tag{12}$$

4) $\tilde{\mathbf{X}}_k$ is finally retracted to the manifold by computing the $\mathbf{Q}$ factor in its reduced QR decomposition, which will be the new $\mathbf{X}_k$.

A difference of the proposed method when compared to other numerical optimization methods for designing Grassmannian constellations, such as [9], [10], is that, in each iteration, we update the position of every codeword of the codebook at (almost) the same time. This is an important aspect of our algorithm that improves the performance in terms of convergence speed and, especially, minimum chordal distance.

Another important aspect of the proposed algorithm is that the value of the parameter μ is adapted using a line-search procedure to speed up convergence. The rate at which we change the value of μ is controlled by the parameter $\alpha \in [1, 1.1]$. After every iteration, if the algorithm does not improve the minimum chordal distance of the codebook, μ is decreased (μ/α). Otherwise, if there is an improvement, μ is increased ($\mu \cdot \alpha$).

The proposed algorithm uses two stopping criteria: a maximum number of iterations and a minimum improvement δ_{min} in the minimum chordal distance of the codebook. After each iteration, the minimum chordal distance of the codebook is recomputed and, if it has not increased more than δ_{min}, the algorithm stops.

The proposed Grassmannian constellation design algorithm is summarized in Algorithm 1.

Algorithm 1: ManOpt

Input: Coherence time T, number of transmit antennas M, codebook size K, initial step-size μ_{ini}, minimum step-size μ_{min}, adaptation rate α, maximum number of iterations N_{max}, minimum required improvement δ_{min}

1 Generate K random subspaces $\langle \mathbf{X} \rangle$ in $\mathbb{G}\left(M, \mathbb{C}^T\right)$
2 Obtain initial minimum chordal distance d_0
3 Initialize $\mu = \mu_{ini}$ and $n = 1$
4 **do**
5   Perform a random permutation of the codebook $\mathcal{C} = \{\mathbf{X}_1, \ldots, \mathbf{X}_K\}$
6   **for** $k = 1 : K$ **do**
7   Find the closest element $\mathbf{X}_j$ to codeword $\mathbf{X}_k$
8   Construct the matrix $\mathbf{\Delta}_{kj} = -\frac{(\mathbf{I}_T - \mathbf{X}_k \mathbf{X}_k^H)\mathbf{X}_j \mathbf{X}_j^H \mathbf{X}_k}{\|(\mathbf{I}_T - \mathbf{X}_k \mathbf{X}_k^H)\mathbf{X}_j \mathbf{X}_j^H \mathbf{X}_k\|_F}$ that yields the best direction to get $\mathbf{X}_k$ far from $\mathbf{X}_j$
9   Move $\mathbf{X}_k$ in the direction defined by $\mathbf{\Delta}_{kj}$ as $\tilde{\mathbf{X}}_k = \mathbf{X}_k + \mu \mathbf{\Delta}_{kj}$
10   Retract $\tilde{\mathbf{X}}_k$ to the manifold by computing the Q factor in its reduced QR decomposition, which will be the new $\mathbf{X}_k$
11   **end for**
12   Obtain minimum chordal distance d_n
13   **if** $d_n > d_{n-1}$ **then**
14   Update codebook $\mathcal{C}$ with the new codewords $\mathbf{X}_k$, $k = 1 : K$
15   **if** $d_n - d_{n-1} < \delta_{min}$ **then**
16   End optimization
17   **end if**
18   Increase step-size $\mu = \alpha\mu$
19   Move to next iteration $n = n + 1$
20   **else**
21   Decrease step-size $\mu = \mu/\alpha$
22   *(note that in this case lines 7 and 8 will not be computed again)
23   **end if**
24 **while** ($n \leq N_{max}$ **and** $\mu \geq \mu_{min}$)

IV. Performance Evaluation

In this section, we provide some simulation examples to assess the performance of the proposed algorithm (ManOpt) in comparison to other numerical optimization algorithms for designing Grassmannian constellations. All the experiments are conducted using MATLAB that runs on a desktop with an Intel(R) Core(TM) i7-6700 3.40GHz CPU and 32GB RAM.

Fig. 1 shows the time (averaged over 50 runs) required by our algorithm (red curves) to generate constellations of sizes ranging from 4 codewords up to 128 codewords for $T = \{4, 6\}$ and $M = 2$. The values of the different ManOpt optimization parameters are $\mu_{ini} = 10^{-1}$, $\mu_{min} = 10^{-4}$, $\alpha = 1.001$, $N_{max} = 200$ and $\delta_{min} = 10^{-5}$.

Besides, we compare it to the time required by the alternating projection (AP) algorithm (blue curves) [13]. This numerical method for finding packings in Grassmannian manifolds basically constructs a matrix (Gram matrix) that needs to have some certain structural and spectral properties. By alternately enforcing the structural condition and then the spectral condition, the algorithm reaches a matrix that satisfies both, from which we can extract a packing.

As we can observe, for both values of T ManOpt clearly outperforms AP. We can also see that an increase in the coherence time T has more impact on our method than in AP, due mainly to the size of the matrices involved in the optimization, which are $T \times T$ for ManOpt and $MK \times MK$ for AP. However, the gap between both methods is significant and for reasonable constellation sizes, ManOpt outperforms AP in terms of speed.

Fig. 2 shows the minimum pairwise chordal distance obtained for different ManOpt constellations (black dots) of sizes ranging from $K = 5$ to $K = 48$ codewords, $T = 4$ and $M = 1$ compared to several theoretical upper bounds that appear in

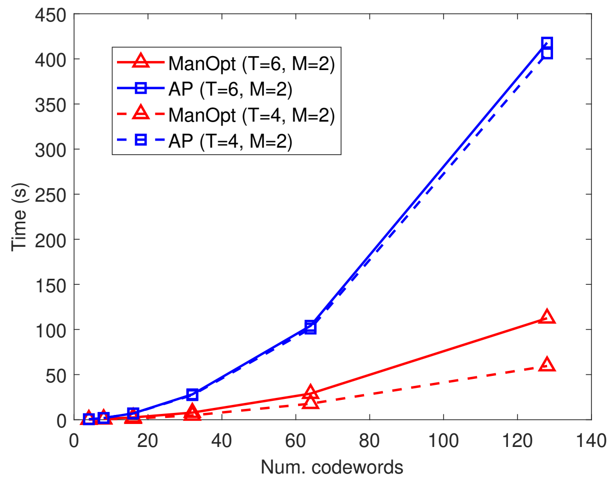

Fig. 1. Time required to generate Grassmannian constellations with AP and the proposed ManOpt method as a function of K (number of codewords).

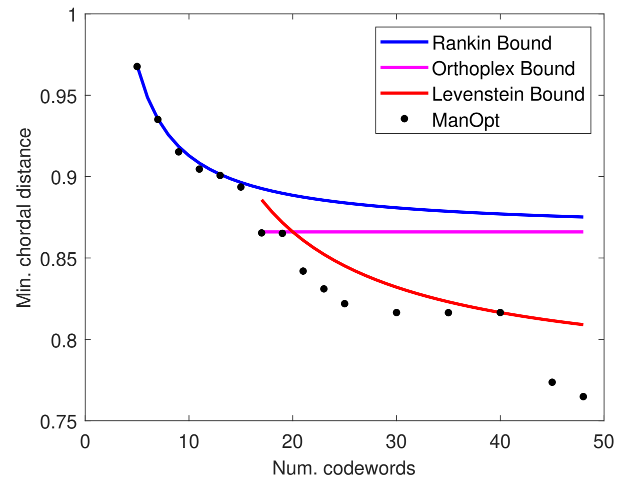

Fig. 2. Minimum chordal distance achieved by the ManOpt algorithm for different constellation sizes with $T = 4$ and $M = 1$.

the literature.

The first bound is the well-known Rankin bound (blue curve), which was first proved on spherical packings in [24]

$$d \leq \sqrt{\frac{T-1}{T}\frac{K}{K-1}} \quad , \quad K > T. \tag{13}$$

The second one is the so-called orthoplex bound (pink curve), which was obtained in [8]

$$d \leq \sqrt{\frac{T-1}{T}} \quad , \quad K > T^2. \tag{14}$$

The last one is the Levenstein bound, whose original proof appears in [25]

$$d \leq \sqrt{\frac{K(T-1)}{(K-T)(T+1)}} \quad , \quad K > T^2. \tag{15}$$

As we can see, our algorithm obtains very good packings that approach the theoretical bounds, especially for $K < 20$ codewords.

To support this idea, we have also compared the best packings known to date, given in [19], with some designs obtained via ManOpt for $T = 7$ and $M = 1$. As we can see in Table I, our method beats the best known packings for $19 < K < 30$. For $K = \{49, 48\}$, the packings in [19] are proved (or conjectured) to be optimal and our algorithm obtains the same chordal distance separation up to sixth decimal place.

Finally, Fig. 3 and 4 shows the SER of ManOpt designs in comparison with other Grassmannian constellations. For these experiments, the parameters used to generate the codebooks are $\mu_{ini} = 10^{-1}$, $\mu_{min} = 10^{-4}$, $\alpha = 1.001$, $N_{max} = 200$ and $\delta_{min} = 10^{-5}$.

Fig. 3 shows the SER curve for the proposed ManOpt constellations (red curves) compared to the AP method (blue curves) for $T = 4$, $M = 1$, $N = \{1, 2\}$ and $K = 64$ codewords. Here we can observe that our constellations slightly outperform the AP designs, besides the design method having lower computational complexity.

TABLE I
BEST FOUND PACKINGS FOR $T = 7$ AND $M = 1$ FOR DIFFERENT CONSTELLATION SIZES K.

Number of codewords	Minimum chordal distance	
(**K**)	Best known packing [19]	ManOpt
19	0.950843	**0.951084**
22	0.946328	**0.946798**
23	0.944544	**0.945397**
24	0.943344	**0.943977**
29	0.937337	**0.939793**
30	0.936184	**0.937866**
48	0.935414	0.935414
49	0.935414	0.935414

Fig. 4 shows the SER curve for ManOpt constellation (red curve) for $T = 4$, $M = 1$, $N = 2$ and $K = 256$ codewords, in comparison with AP and two structured Grassmannian constellation designs: exp-map and cube-split. The first one is built from space-time codes for coherent systems via a non linear map (parameterization), which is called the exponential map [11]. The second one is generated by partitioning the Grassmannian of lines into a collection of bent hypercubes and defining a mapping onto each of these bent hypercubes such that the resulting symbols are approximately uniformly distributed on the Grassmannian. As we can see, our design clearly beats the exp-map and cube-split algorithms and slightly outperforms the AP constellation.

V. CONCLUSIONS

A fast algorithm for designing Grassmannian constellations for noncoherent communications, named ManOpt, is proposed in this paper, which is based on a gradient ascent approach with adaptive step-size that operates directly on the Grassmann manifold to maximize the minimum pairwise chordal distance.

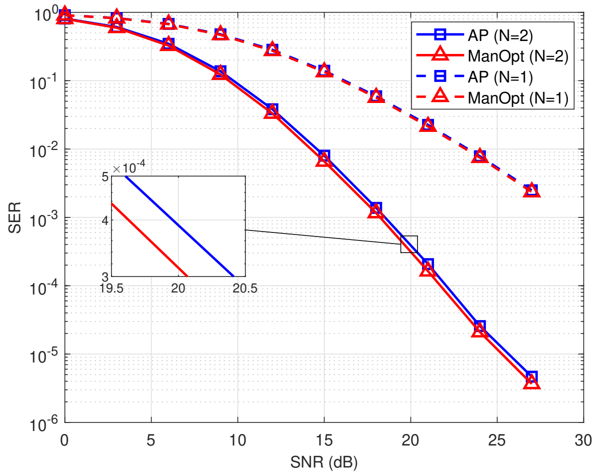

Fig. 3. SER curves for $K = 64$ codewords, $T = 4$ and $M = 2$.

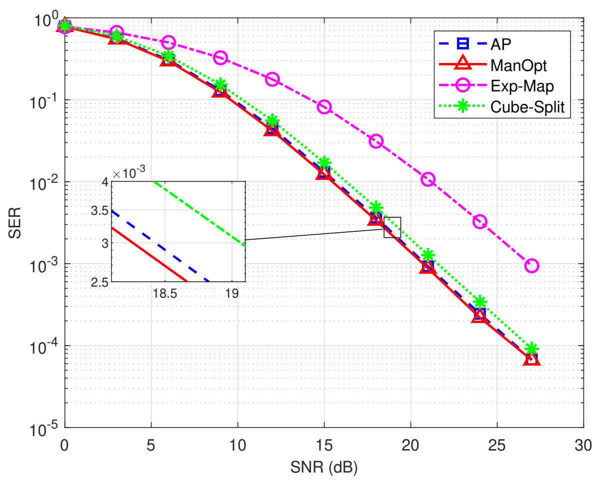

Fig. 4. SER curves for $K = 256$ codewords, $T = 4$, $M = 1$ and $N = 2$.

The proposed algorithm outperforms other numerical optimization methods for designing Grassmannian constellations, such as alternating projection, in terms of convergence speed, minimum pairwise chordal distance and SER. It also offers superior performance when compared to structured constellation design methods, such as exp-map or cube-split, in terms of minimum pairwise chordal distance and SER.

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