Optimal Estimation of a Class of Chaotic Signals

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Abstract

Signals generated by iterating nonlinear maps are highly attractive in a wide range of signal processing applications. Among the different possible one-dimensional chaotic systems, an important class is composed of the so-called skew tent maps. In this paper, an algorithm for the optimal estimation of this class of signals in the presence of noise is developed based on the Maximum Likelihood (ML) method. The resulting algorithm is quite demanding computationally, so suitable suboptimal schemes are proposed that show good performance at a much reduced computational cost. Computer simulations are included, and the performance of the different approaches compared with the associated Cramer-Rao Lower Bound (CRLB).

Keywords

Chaos, Skew Tent Maps, Maximum Likelihood Estimation.

1. Introduction

Chaotic signals, signals obtained by iterating a non linear dynamical system in chaotic state, have become an active field of research [1]. Classical signal processing techniques do not perform adequately for this class of signals that show a noise-like behavior, although deterministic in nature. Therefore, it is important to develop new algorithms suited for this type of signals. In particular, there is a need for robust and efficient algorithms for the estimation of these signals in noise.

Estimation of chaotic signals in noise has been addressed in numerous papers. In [2] the performance of the ML estimator for chaotic signals generated by one-dimensional maps is analyzed. The estimator is found to be inconsistent, so the asymptotic distribution for large data records is invalid. However, for a high Signal to Noise Ratio (SNR) the ML estimator is asymptotically unbiased and attains the CRLB. In [3] an algorithm for chaotic signal estimation based on the connection between the symbolic sequence and the initial condition is presented, which is shown to attain the CRLB at high SNR. This approach is closely related with the *halving method* presented in [4], where a dynamical programming ML estimator is proposed. Finally, in [5] a recursive implementation of the ML estimator for chaotic signals generated by tent maps is derived. However, no ML estimator has been derived for skew tent maps. Chaotic signals generated by iterating skew tent maps have been applied in communications [6], and their spectral properties have been found to be identical to those of a stochastic first-order autoregressive (AR) process [7].

In this paper we develop an algorithm for ML estimation of chaotic signals whose dynamics are governed by the skew tent map. The resulting method is quite demanding computationally, so two suboptimal approaches are also proposed that reduce considerably the computational cost, maintaining a good performance nevertheless. All the three estimators are inconsistent, but attain the CRLB for high SNR.

2. Skew tent maps

The signals that we consider in this work are generated according to

$$x[n+1] = F\big(x[n]\big) \tag{1}$$

where F is a mapping of the unit interval into itself ([0,1]→[0,1]) given by

$$F(x) = \begin{cases} x/a & 0 \le x \le a \\ (1-x)/(1-a) & a < x \le 1 \end{cases} \tag{2}$$

which is called skew tent map, with known parameter a, where $0<a<1$. When $a=0.5$, F becomes the tent map.

The phase space of non-linear maps can be divided in a collection of non-overlapping regions. This process

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is known as *partition of the phase space* [3]. If a symbol from a known alphabet is assigned to each of the regions, the dynamics of the map may be characterized by following the different regions that the map visits during its dynamical evolution. The analysis of chaotic signals using these sequences of symbols (known as itineraries) is covered by the field of *symbolic dynamics* [3]. In the particular case of skew tent maps, we divide the phase space in two regions $E_1=[0,a]$ and $E_2=[a,1]$, and we associate a symbol $s[n]$ to each $x[n]$ according to

$$s[n] = sign\left(x[n]-a\right) \tag{3}$$

The Function F is generally non-invertible, as it has two inverse images given by

$$F^{-1}_{\mathbf{s}[n]}(x[n+1]) = \begin{cases} ax[n+1] & s[n]=-1 \\ 1-(1-a)x[n+1] & s[n]=1 \end{cases} \tag{4}$$

However, with s[n] known, a useful representation of the chaotic signal may be obtained by using the inverse mapping (4). Any element of the sequence may be obtained by backward iteration from $x[N]$

$$x[n] = F^{-1}_{s[n]} \circ F^{-1}_{s[n+1]} \circ \ldots \circ F^{-1}_{s[N-1]}\left(x[N]\right) \tag{5}$$

with $s[n]$ given by (3). Thus, knowing the itinerary and any value of the signal, we can reconstruct the full sequence. We will denote $\mathbf{s}=[s[0], s[1], ..., s[N-1]]$ the itinerary associated with a certain chaotic signal. We will define R_i the region of the phase space comprised between x^i_{min} and x^i_{max}, and associated with a certain itinerary $\mathbf{s}_i$. This is the region where the initial condition $x[0]$ must lie to generate sequences with itinerary $\mathbf{s}_i$. We will define as well an indicator function (sometimes called characteristic function)

$$c_i(x) = \begin{cases} 1 & x \in R_i \\ 0 & x \notin R_i \end{cases} \tag{6}$$

As the skew tent maps are onto, all the itineraries are possible, and there are 2^N regions associated with the 2^N possible itineraries of length N. The easiest way of obtaining the limits of the region R_i is iterating backwards from a using (4), leading to what is called a natural partition of the phase space [8].

3. ML estimation of chaotic sequences

The problem we are considering is the estimation of a chaotic sequence from N+1 noisy observations

$$y[0], y[1], \ldots, y[N] \tag{7}$$

obtained according to

$$y[n] = x[n] + n[n] \tag{8}$$

where $n[n]$ is a stationary, zero mean, white Gaussian noise sequence with variance s^2, and $x[n]$ is a chaotic sequence generated using (2), by iterating some unknown $x[0]\in[0,1]$ according to (1), for some known parameter $0<a<1$.
As the sequence of observations is a collection of independent Gaussian random variables with equal variance, ML estimation produces the initial condition that minimizes the Mean Square Error (MSE)

$$J(x[0]) = \sum_{n=0}^{N}\left(y[n] - F^n\left(x[0]\right)\right)^2 \tag{9}$$

where F^n is the n-fold composition of F. A direct minimization of (9) using, for example, gradient descent techniques, cannot be applied to this problem, because the error surface is highly irregular with many local minima [5]. It seems that the minimization of such an error surface demands exhaustive search solutions. This is not the case, however, if we take a closer look to the MSE, drawing the different regions R_i. An example is shown in Figure 1, where the MSE for $x[0]=0.7218$, $N=6$ and $a=0.8$ is plotted in the range [0.68,0.75]. It is clear that the MSE curve is quadratic within the limits of each region, so there is a unique minimum in each region.

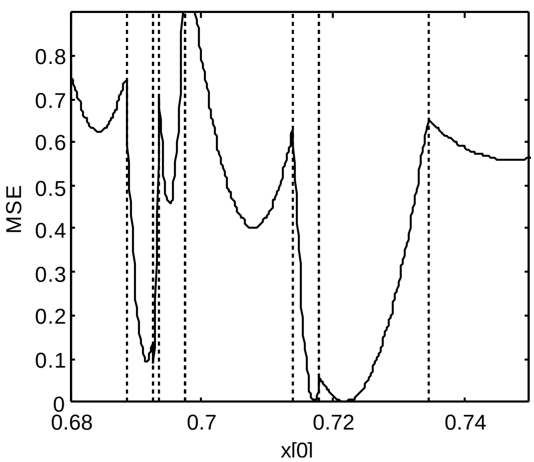

Figure 1. Error surface for $x[0]=0.7218$, $N=6$ and $a=0.8$ in the range [0.68,0.75]. The dashed lines mark the limits of the regions R_i.

To obtain these minima we have to find a closed form expression for the n-fold composition of F. It is easy to verify that $F(x[n])$ can be expressed as

$$x[n+1] = F(x[n]) = \frac{x[n](1\text{-}2a)\text{-}\left|x[n]\text{-}a\right| + a}{b} \quad (10)$$

where b=2a(1-a), and using the symbol $s[n]$

$$F(x[n]) = \frac{x[n](1\text{-}2a - s[n]) + a(1+s[n])}{b} \quad (11)$$

Finally, the n-fold composition is given by

$$F_{\mathbf{s}}^{n}\left(x[0]\right) = a\sum_{i=0}^{n-1} b^{i-n}(1+s[i])S_{n-i-1}^{n} + b^{-n}S_{n}^{n}[n]x[0] \quad (12)$$

where $S_0^n = 1$, and

$$S_i^n = \prod_{j=n-i}^{n-1}\left(1-2a-s[j]\right) \qquad 1 \le i \le n \quad (13)$$

Using (12) we can express (9) in a region R_i as

$$J_i(x[0]) = \sum_{n=0}^{N}\left(y[n] - F_{\mathbf{s}_i}^{n}\left(x[0]\right)\right)^2 \quad (14)$$

and the MSE may be expressed as

$$J(x[0]) = \sum_{i=1}^{2^N} c_i(x[0])J_i(x[0]) \quad (15)$$

Taking into account (12) and (14) it is easy to conclude that the cost function is quadratic within the limits of each region. Depending on the noise level the minimum of each $J_i(x[0])$ may fall in or outside the region R_i. Differentiating (13) and solving for the unique minimum we obtain an estimate of $x[0]$ for a given itinerary

$$\hat{x}_i[0] = \frac{\sum_{n=0}^{N} b^{-n}S_n^n g[n]}{\sum_{n=0}^{N}\left(b^{-n}S_n^n\right)^2} \quad (16)$$

where

$$g[n] = y[n] - a\sum_{i=0}^{n-1} b^{i-n}(1+s[i])S_{n-i-1}^{n} \quad (17)$$

This is the ML estimate of $x[0]$ for the given itinerary only if $\hat{x}_i[0] \in R_i$. Otherwise the minimum of $J(x[0])$ in the region R_i is given by the closest value in R_i to $\hat{x}_i[0]$. Taking all this into account, the ML estimate with a known itinerary $\mathbf{s}_i$ is given by

$$\hat{x}_{ML}^{i}[0] = \begin{cases} \hat{x}_i[0] & \hat{x}_i[0] \in R_i \\ x_{min}^{i} & \hat{x}_i[0] < x_{min}^{i} \\ x_{max}^{i} & \hat{x}_i[0] > x_{max}^{i} \end{cases} \quad (18)$$

And finally, the ML estimate of $x[0]$ is associated with the itinerary that produces the minimum MSE

$$k = \arg\min_i (J(\hat{x}_{ML}^{i}[0])) \quad (19)$$

and the ML estimate is

$$\hat{x}_{ML}[0] = \hat{x}_{ML}^{k}[0] \quad (20)$$

4. Suboptimal Approaches

The method derived in the previous section demands the computation of 2^N estimates, and comparing the MSE produced by all of them to obtain the final ML estimate. This approach may become quite costly computationally for moderate sized records. In this Section we will introduce two suboptimal approaches that achieve a close to optimum performance, while reducing the computational cost. Both of them are based on estimating the itinerary and then applying (16) and (18) to obtain the final estimate of $x[0]$.

The first approach, that we will call hard-censoring (H-C) ML, uses the same method proposed in [3] and obtains the itinerary from the noisy observations

$$\hat{s}[n] = sign(y[n]\text{-}a) \quad (21)$$

and using (16) and (18) we obtain the estimate. It should be noted that this approach produces the ML estimate when (21) gives the ML estimate of the itinerary, as opposed to [3].

The second approach, that we will call recursive ML, is equivalent to the one proposed in [5], and is optimal for the tent map (a=0.5). It obtains ML estimates for registers of increasing lengths from n=1 to N, and estimates $s[n]$ as

$$\hat{s}[n] = sign\left(\hat{x}[n \mid n]\text{-}a\right) \quad (22)$$

where $\hat{x}[n \mid n]$ is the ML estimate of $x[n]$ using the first n samples of the register. The algorithm is pursued recursively until the itinerary is completely estimated. Then (16) and (18) are used to obtain the final estimate. Once again, when (22) gives the ML estimate of the itinerary, this approach produces the ML estimate of $x[0]$.

5. Computer Simulations

In this section we analyze the performance of the optimal ML estimator in comparison with the suboptimal estimators. We consider a skew tent map with a=0.9 and N=6. 999 initial conditions uniformly distributed from 0.001 to 0.999 have been selected. Table 1 shows the Mean Square Error (MSE) for the three estimators obtained by Monte Carlo simulations by averaging 1000 cases for each initial condition and SNR.

Table 1 – Average MSE (in dB) of the three estimators

	-10·log10(MSE)		
SNR	ML	H-C ML	Recursive ML
0	14.7	15.4	15.1
5	18.7	18.7	18.8
10	24.3	22.6	23.0
15	30.4	27.0	27.9
20	36.8	31.7	33.3
25	43.3	38.0	40.7
60	89.0	88.4	88.6

The ML estimator shows the best performance, is asymptotically unbiased and attains the CRLB at high SNR. The H-C ML estimator shows the worst performance of the three, while the recursive ML estimator lies in between the other two. An example of an MSE curve is shown in Figure 2, for $x[0]$=0.833, N=6 and a=0.9.

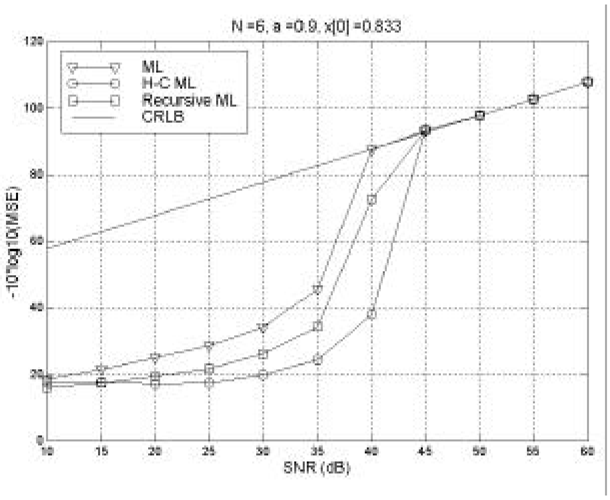

Figure 2. MSE of the three estimators proposed for $x[0]$=0.833, N=6 and a=0.9.

6. Conclusions

In this paper we have developed an ML estimator for chaotic signals generated by iterating skew tent maps and observed in noise. The estimator is inconsistent but achieves the CRLB for high SNR. As the optimal estimator demands a high computational cost for moderate sized data records, two suboptimal approaches have been proposed. The first one obtains the itinerary directly from the noisy data record, while the second one obtains forward ML estimates of $x[n]$ recursively from its previous samples, and estimates $s[n]$ as the sign of these forward estimates. These two approaches reduce considerably the computational cost and show good performance, achieving the CRLB for high SNR.

Future research lines include the development of ML estimators for other piecewise-linear maps, the search for effective algorithms for optimal estimation of the itinerary and the derivation of Bayesian estimators of chaotic signals.

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