# Source Enumeration in Non-White Noise and Small Sample Size via Subspace Averaging

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***Abstract*—This paper addresses the problem of source enumeration by an array of sensors in the challenging conditions of: i) large uniform arrays with few snapshots, and ii) non-white or spatially correlated noises with arbitrary correlation. To solve this problem, we combine a subspace averaging (SA) technique, recently proposed for the case of independent and identically distributed (i.i.d.) noises, with a majority vote approach. The number of sources is detected for increasing dimensions of the SA technique and then a majority vote is applied to determine the final estimate. As illustrated by some simulation examples, this simple modification makes SA a very robust method of enumerating sources in these challenging scenarios.**

***Index Terms*—Array processing, model order estimation, source enumeration, subspace averaging.**

## I. Introduction

To estimate the number of sources received by an array of sensors, which is referred to as source enumeration here, is a classical problem in statistical signal processing with many practical applications ranging from radar and wireless communications to biomedical and geophysical signal processing [1], [2]. Numerous approaches to this problem have been proposed in the last decades based on information theoretic criteria such as Akaike's information criterion (AIC) [3], minimum description length (MDL) criterion [4], [5], and Bayesian information criterion (BIC) [6], all of which are functions of the eigenvalues of the sample covariance matrix. Given a sufficiently large sample size (at least several times the number of sensors), and assuming that the noise is spatially white and Gaussian, these methods perform satisfactorily and provide accurate estimates for the number of sources. However, their performance degrades drastically when the number of snapshots, $N$, is significantly smaller than the number of sensors, $M$; or when the noise is non-white. This paper focuses on this challenging scenario for large-scale arrays.

Based on recent results from random matrix theory, several methods have been proposed for source enumeration in the small sample regime [7]–[9]. Nevertheless, all these methods also assume white noise and hence provide in general poor results in spatially correlated noise.

When the noise is spatially correlated with an arbitrary unknown covariance matrix, source enumeration has been considered under different assumptions on the array geometry and the temporal correlation of the noise in [10]–[13]. In [10] the authors assume that two well-separated sensor arrays are available and thus the noise spatial covariance matrix is block diagonal. The resulting test is based on the canonical correlations estimated from the sample coherence matrix. The method in [13] assumes that the signal is received by a uniform linear array (ULA) for which a property called shift invariance holds [14]–[16], and propose an ad-hoc test based on the elements of the rotation matrix that relates the signal subspaces extracted from the two subarrays. These methods, however, require accurate estimates of the sample covariance or the coherence matrix and therefore their performance degrades when only a few snapshots are available. To alleviate this problem, [17] applies a principal component analysis (PCA) rank-reduction preprocessing step before applying the Bartlett-Lawley test [18], [19].

When the noise covariance matrix is diagonal with unknown elements, estimating the number of sources is equivalent to estimating the number of common factors in a multivariate factor analysis problem, for which several tests have been proposed in the statistics literature [20], [21]. Algorithms to maximize the likelihood function for this problem can be found in [22], [23]. But again these methods perform poorly in the small sample regime.

In summary, source enumeration in non-white noise with high-dimensional data (i.e., large arrays) and few snapshots is still a challenging problem for which no method has been found in the literature that works robustly under different models for the noise covariance matrix. To tackle this problem, in this paper we present an improved version of the subspace averaging (SA) technique for order estimation originally developed for the case of white noise in [24]–[26]. Under non-white noise scenarios, the original SA criterion is very sensitive with the chosen dimension of extracted subspaces $d$. To increase the robustness of the SA method, in this paper we apply the SA order estimation rule for increasing values of $d$ and then apply a majority vote to determine the final estimate of the number of sources. This simple modification, makes SA a very robust method of enumerating sources in large linear arrays under conditions of low sample support and under different models for the noise covariance matrix.

## II. Problem Statement

Let us consider $K$ narrowband signals impinging on a large, uniform, half-wavelength linear array with $M$ antennas as

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depicted in Fig. 1. The received signal is

$$\mathbf{x}[n] = [\mathbf{a}(\theta_1) \quad \cdots \quad \mathbf{a}(\theta_K)]\, \mathbf{s}[n] + \mathbf{e}[n] = \mathbf{A}\mathbf{s}[n] + \mathbf{e}[n], \quad (1)$$

where $\mathbf{a}(\theta_k) = \begin{bmatrix}1 & e^{-j\theta_k} & e^{-j\theta_k(M-1)}\end{bmatrix}^T$ is the $M \times 1$ complex array response for the $k$th source, $s_k$, with unknown direction-of-arrival (DOA) $\theta_k$ (electrical angles). The signals are assumed to be uncorrelated and are modeled as $\mathbf{s}[n] \sim \mathcal{CN}_K(\mathbf{0}, \sigma_s^2\mathbf{I})$. The steering matrix is $\mathbf{A} \in \mathbb{C}^{M\times K}$. From the signal model (1), the covariance matrix is

$$\mathbf{R} = E\left[\mathbf{x}[n]\mathbf{x}^H[n]\right] = \sigma_s^2 \mathbf{A}\mathbf{A}^H + \mathbf{\Sigma}, \quad (2)$$

where $\mathbf{\Sigma}$ is $M \times M$ noise covariance matrix. In this paper we consider the following non-white noise models:

- **Model 1**: Uncorrelated noises across antennas so that the noise covariance matrix is diagonal with unknown elements along its diagonal

$$\mathbf{\Sigma} = \mathrm{diag}\left(\sigma_1^2, \sigma_2^2, \ldots, \sigma_M^2\right).$$

  We assume that the noise variances at every antenna are independent random variables uniformly distributed as $\sigma_i^2 \sim \mathcal{U}[\sigma^2(1-\epsilon), \sigma^2(1+\epsilon)]$, where $\sigma^2$ is a common noise variance and $\epsilon$ allows us to control the non-whiteness of the noise. Notice that for $\epsilon = 0$ the noise is white with covariance matrix $\mathbf{\Sigma} = \sigma^2\mathbf{I}$.
- **Model 2**: The noise has an arbitrary psd covariance matrix $\mathbf{\Sigma} \succ 0$.

Let $\mathbf{A}_s$ be the $L \times K$ subarray matrix with rows $s, \ldots, s+L-1$ extracted from the steering matrix $\mathbf{A}$ (see Fig. 1). Then, from (1) it is readily verified that

$$\mathbf{A}_s \,\mathrm{diag}(e^{-j\theta_1}, \ldots, e^{-j\theta_K}) = \mathbf{A}_{s+1},$$

for $s = 1, \ldots, M-L+1$, which is the so-called shift invariance property [14]–[16], [27]. In this way, $\mathbf{A}_s$ and $\mathbf{A}_{s+1}$ are related by a nonsingular rotation matrix, and therefore span the same subspace. It is also possible to show that the $K$ principal eigenvectors of $\mathbf{R}$ are also shift-invariant [15], [16].

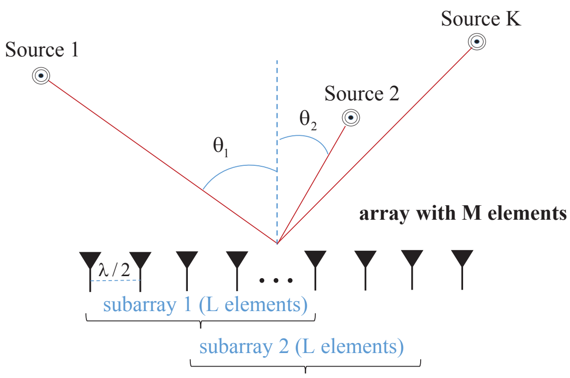


Fig. 1. $L$-dimensional subarrays extracted from a uniform linear array (ULA) with $M > L$ elements.

We assume there are $N$ snapshots collected in the data matrix matrix $\mathbf{X} = \begin{bmatrix}\mathbf{x}[1] & \ldots & \mathbf{x}[N]\end{bmatrix}$, and the sample covariance matrix is

$$\hat{\mathbf{R}} = \frac{1}{N}\sum_{n=1}^{N} \mathbf{x}[n]\mathbf{x}^H[n] = \frac{1}{N}\mathbf{X}\mathbf{X}^H. \quad (3)$$

The source enumeration problem consists of estimating $K$ from $\mathbf{X}$ or $\hat{\mathbf{R}}$. Finally, we also assume that the array is composed of a large number of antenna elements, and that the number of snapshots is possibly smaller than the number of antennas, that is, $N < M$.

## III. Prior Work

In this section we review some representative methods for order estimation in non-white noises and/or with small sample support.

### A. *Methods suitable for Model 1*

**MDL for Model 1**: The MDL has the following general expression

$$\hat{k}_{MDL} = \underset{k=0,\ldots,M-1}{\mathrm{argmin}} \left\{ -\log f(\mathbf{X}|\hat{\mathbf{R}}_{ML}) + \frac{1}{2}\nu_k \log N \right\}, \quad (4)$$

where $\hat{\mathbf{R}}_{ML}$ denotes the maximum likelihood (ML) estimate for a covariance matrix with the required structure (low-rank + diagonal) for a fixed order $k$, and $\nu_k = M + k(2M-k)$ is the number of free-adjusted real parameters [28]. The ML estimate $\hat{\mathbf{R}}_{ML}$ under noise Model 1 does not admit a closed-form expression, but we can resort to any of the iterative algorithms proposed in [23], [29]. Notice that here we are not exploiting the shift invariance property of the steering matrix, which is also reflected in the number of free parameters $\nu_k$. An iterative ML algorithm that estimates the directions of arrival and hence exploits the structure of the steering matrix is described in [22].

**LS-MDL** and **BIC**: The standard MDL method proposed by Wax and Kailath [5] under the assumption of white noise is

$$\hat{k}_{MDL} = \underset{0\le k\le M-1}{\mathrm{argmin}} \; (M-k)N \log\left(\frac{a(k)}{g(k)}\right) + \frac{1}{2}k(2M-k)\log N, \quad (5)$$

where $a(k)$ and $g(k)$ are the geometric and the arithmetic mean, respectively, of the $M-k$ smallest eigenvalues of $\hat{\mathbf{R}}$. The linear shrinkage MDL (LS-MDL) method proposed by Huang and So in [8] replaces the noise eigenvalues in the MDL criterion by a linear shrinkage. On the other hand, the classical BIC criterion [6] was adapted for the small-sample regime in [9]. For details about these methods, the reader is referred to [8], [9].

### B. *Methods suitable for Model 2*

Under Model 2 the array noise is spatially correlated in an arbitrary way. Therefore, without imposing further structure in the problem, it is not possible to use MDL or any other information theoretic criteria. The only alternatives in the literature either assume the availability of two well-separated subarrays such that the noise covariance matrix can be assumed to be block-diagonal [10], [17], or exploit the shift invariance property of the signal subspace [13]. In the following, we briefly review both approaches.

**CCA**: Let us divide the array into two equal-sized non-overlapping subarrays of dimension $M/2$ (we assume $M$ even wlog). The composite vector for the received signal is written as $\mathbf{x}[n] = (\mathbf{x}_1[n]^T, \mathbf{x}_2[n]^T)^T$, where $\mathbf{x}_1[n]$ and $\mathbf{x}_2[n]$ are the signals received at each subarray. Assuming now that the noise vectors at the two subarrays are uncorrelated, the noise covariance matrix is block diagonal $\mathbf{\Sigma} = \text{blkdiag}(\mathbf{\Sigma}_1, \mathbf{\Sigma}_2)$. Under these assumptions, it was shown in [10] (see also [30]) that the maximum of the log-likelihood function in (4) can be written as

$$-\log f(\mathbf{X}|\hat{\mathbf{R}}_{ML}) = -N \sum_{i=k+1}^{M/2} \log(1 - k_i^2), \tag{6}$$

where $k_i$ is the $i$-th sample canonical correlation, that is the $i$-th eigenvalue of the coherence matrix $\hat{\mathbf{C}} = \hat{\mathbf{R}}_{x_1x_1}^{-1/2}\hat{\mathbf{R}}_{x_1x_2}\hat{\mathbf{R}}_{x_2x_2}^{-1/2}$. On the other hand, the number of free parameters for this noise model is $\nu_k = 2k(M - k + 1/2)$. Admittedly, this MDL criterion is obtained for a structured block-diagonal noise covariance matrix, which is different from Model 2. Nevertheless, it is used for comparison in this paper. This canonical correlation analysis (CCA) based criterion was combined with a PCA rank-reduction preprocessing step in [17], to improve its performance in the small sample regime.

**VTRS:** Jiang and Ingram proposed in [13] a method that uses the variance of transformed rotational submatrix (VTRS) as a criterion for the detection of number of sources. The VTRS criterion is

$$\hat{k}_{VTRS} = \underset{1 \le k \le M-1}{\text{argmin}} \frac{||\mathbf{\Delta}_k||^2}{(M-k-1)k},$$

where $||\cdot||^2$ denotes the squared Frobenius norm and

$$\mathbf{\Delta}_k = \begin{bmatrix} \Psi_{k+1,1} & \Psi_{k+1,2} & \cdots & \Psi_{k+1,k} \\ \Psi_{k+2,1} & \Psi_{k+2,2} & \cdots & \Psi_{k+2,2} \\ \vdots & \vdots & \ddots & \vdots \\ \Psi_{M-1,1} & \Psi_{M-1,2} & \cdots & \Psi_{M-1,k} \end{bmatrix}$$

is a submatrix of $\mathbf{\Psi}$, which is given as $\mathbf{E_y} = \mathbf{E_x}\mathbf{\Psi}$, where the matrices $\mathbf{E_x}$ and $\mathbf{E_y}$ contain the first $M-1$ and last $M-1$ rows of the eigenvectors of $\hat{\mathbf{R}}$, respectively.

The VTRS criterion exploits the shift invariance property of the array and can be used with arbitrarily correlated noises. However, it is not designed to work with small sample support.

## IV. Order Estimation by Subspace Averaging

A subspace averaging (SA) technique to estimate the dimension of a central subspace from a collection of subspaces has been proposed in [24]. Exploiting the shift invariance property of uniform linear arrays, SA was later used in [25], [26] to determine the number of sources in large arrays under conditions of low sample support. The SA method provides competitive results when the noises are i.i.d., but its performance decreases under spatially correlated noises. In this section, we first review the original SA technique and then present a simple modification to accurately detect the number of sources in non-white noises and with a small sample size.

### *A. SA criterion*

Let $\mathbf{x}_s[n]$ be the $n$-th snapshot of the $s$-th subarray in Fig 1, and let $\hat{\mathbf{R}}_s$ be the $L \times L$ sample covariance, which corresponds to a submatrix of the full sample covariance. Since there are $M$ sensors and we extract $L$-dimensional subarrays, there are $S = M - L + 1$ different submatrices $\hat{\mathbf{R}}_s$, $s = 1, \ldots, S$.

Due to the shift invariance property of uniform linear arrays the noiseless signal subspaces of the theoretical $\mathbf{R}_s$ are identical. The SA method extracts for each subarray a subspace formed by the $d$ largest eigenvectors of $\mathbf{R}_s$. A unitary basis for this subspace is denoted as $\mathbf{V}_s \in \mathbb{C}^{L \times d}$, and the orthogonal projection matrix onto the subspace is $\mathbf{P}_s = \mathbf{V}_s\mathbf{V}_s^H$. The SA order determination rule first computes the average projection matrix

$$\overline{\mathbf{P}} = \frac{1}{S}\sum_{s=1}^{S} \mathbf{P}_s, \tag{7}$$

and its eigenvalues $1 \ge k_1 \ge k_2 \ge \ldots \ge k_L$. Finally, the number of sources is determined as the number of eigenvalues larger than 1/2.

### *B. A majority vote approach*

A limitation of the SA method is that its performance is sensitive to the dimension of extracted subspaces $d$. This problem becomes more important under spatially correlated noises because the directions of the extracted subspaces that correspond to the noise subspace tend to be correlated for consecutive subarrays.

Fig. 2 illustrates the influence of $d$ in the order estimate provided by SA when the noise is spatially correlated in an arbitrary way (Model 2). In this scenario we have $M = 100$ antennas, $N = 50$ snapshots and different number of sources separated $\Delta_\theta = 10°$. This example suggests a simple procedure for estimating $K$. First, the SA order estimation rule is applied for a sequence of increasing values of $d$, $1 \le d \le d_{max}$, where $d_{max}$ is an overestimate of the maximum number of sources that we expect in our problem. The final estimate is obtained by majority vote. This simple modification makes the SA method a robust source enumeration technique suitable for noises with different spatial correlation models and with very few snapshots. In comparison to the original SA method for white noise [25], the computational cost is increased by a factor of $d_{max}$. A summary of the final algorithm is shown in Algorithm 1.

## V. Simulation Results

In this section, we compare the performance of the SA method in scenarios with non-white noise and small sample support. For comparison we we use the following methods: LS-MDL [8], BIC for large-scale arrays [9], the MDL for

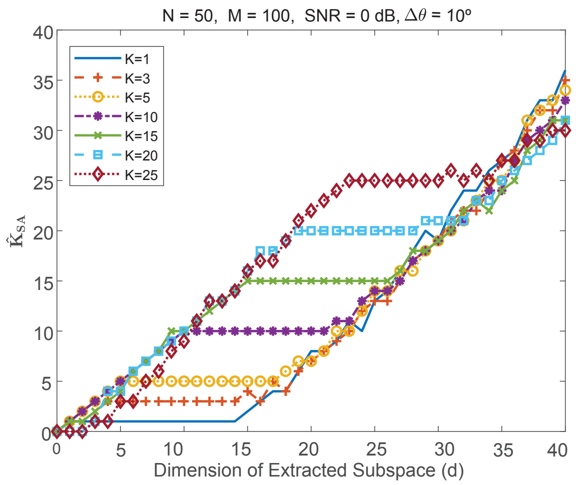


Fig. 2. Estimated number of sources for SA method vs $d$ (dimension of extracted subspaces) in non-white noise (noise Model 2) for $N = 50$, $M = 100$, $\Delta_\theta = 10°$ and SNR $= 0$ dB.

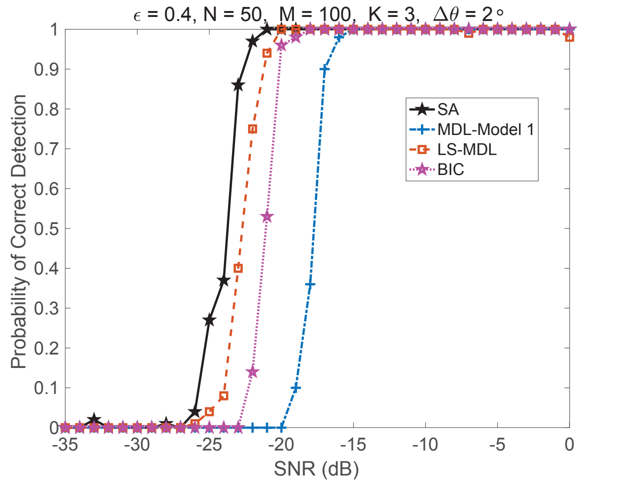


Fig. 3. Probability of correct detection vs SNR for Model 1 when $\epsilon = 0.4$, $N = 50$, $M = 100$, $K = 3$ and $\Delta_\theta = 2°$.

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**Algorithm 1:** Subspace Averaging Criterion.

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**Input:** $\hat{\mathbf{R}}$, $L$;
**Output:** Order estimate $\hat{k}_{SA}$
**Initialization**: $\boldsymbol{K} = \mathbf{0}_{d_{max} \times 1}$
**for** $t = 1 \dots d_{max}$ **do**
  **for** $s = 1, \dots, S$ **do**
    Extract $\hat{\mathbf{R}}_s$ from $\hat{\mathbf{R}}$ and obtain $\hat{\mathbf{R}}_s = \mathbf{U}_s \boldsymbol{\Sigma}_s \mathbf{U}_s^H$
    Compute the projection matrices $\mathbf{P}_{st} = \mathbf{V}_{st} \mathbf{V}_{st}^H$, where $\mathbf{V}_{st} = [\mathbf{u}_{s,1}, \dots, \mathbf{u}_{s,t}]$
  Compute $\overline{\mathbf{P}}$ and its eigenvalues $(k_1, \dots, k_L)$
  Estimate $\hat{k}$ as the number of eigenvalues of $\overline{\mathbf{P}}$ larger than 1/2
  $\boldsymbol{K}(t) = \hat{k}$
Select $\hat{k}_{SA}$ as the majority decision from the collection of all estimated $\hat{k}$ in $\boldsymbol{K}$

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noise Model 1 (denoted here as MDL-Model 1), the PCA-CCA method in [17], and the VTRS method in [13]. These methods have been briefly described in Section III.

We consider a scenario with $K$ narrowband incoherent unit-power signals and DOAs separated $\Delta_\theta$ impinging on an uniform linear array with $M$ antennas and half-wavelength element separation. For the SA method, we use $L = M - 5$ as the subarray size, and $d_{max} = M/5$. The signal-to-noise-ratio (SNR) is $\text{SNR} = 10 \log \frac{K}{\text{tr}(\boldsymbol{\Sigma})}$, where $\text{tr}()$ denotes trace. The results shown in the curves represent the average of 100 independent simulations.

**Model 1**: In the first experiment we consider an array with $M = 100$ antennas, $K = 3$ sources with sources separated $\Delta_\theta = 2°$, and $N = 50$ snapshots. The noise is drawn according to Model 1 with $\epsilon = 0.4$. Fig. 3 shows the probability of correct detection for this example. Although the LS-MDL and the BIC methods assume i.i.d. noises, in this scenario with very few snapshots their performance is rather robust against a mismatched model. Nevertheless, the SA technique outperforms the other methods.

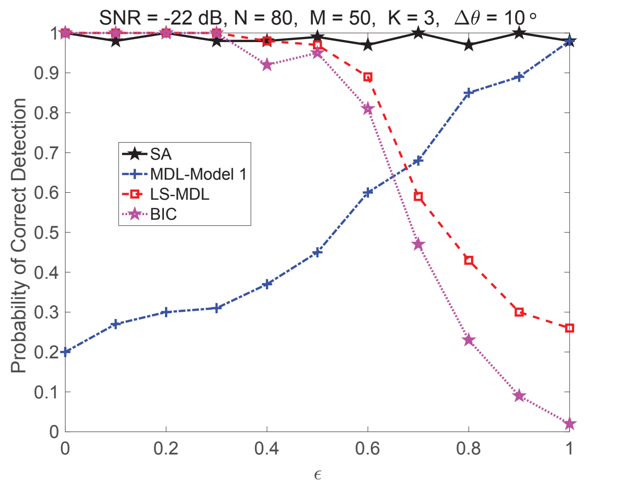


Fig. 4. Probability of correct detection vs $\epsilon$ for Model 1 when $N = 80$, $M = 50$, $K = 3$, SNR $= -22$ dB and $\Delta_\theta = 10°$.

Fig. 4 shows the results when $\epsilon$ varies (recall that for $\epsilon = 0$ noise is spatially white). For large values of $\epsilon$, the performance of both the LS-MDL and the BIC criteria degrade, whereas the MDL specifically designed for this model behaves better. Interestingly, the SA method provides accurate estimates across the whole range of $\epsilon$.

**Model 2**: We now consider that the noise is correlated in an arbitrary way (Model 2) and compare the performance of the SA method with the PCA-CCA and the VRTS methods. We consider an array with $M = 100$ sensors, $K = 4$ sources separated $\Delta_\theta = 10°$, the SNR is fixed, and the number of snapshots varies between $N = 20$ and $N = 400$. We observe that the VTRS method requires at least $N = 300$ snapshots to provide accurate estimates. For the PCA-CCA method we use two equal-sized subarrays of 50 antennas each. The assumed noise model for the PCA-CCA criterion is mismatched to Model 2, so, not surprisingly, its estimates are not always good. The MDL for Model 1 is also mismatched to the true model, however it performs well when $N > 150$. Finally, the proposed SA method with majority vote detects the correct number of sources with high accuracy with only $N = 50$ snapshots.

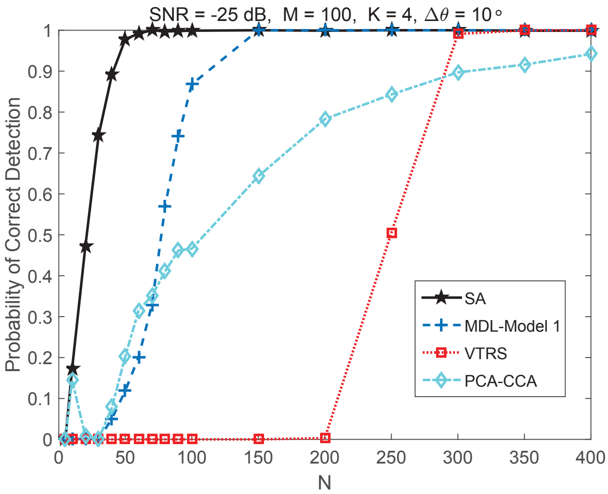


Fig. 5. Probability of correct detection vs snapshots for Model 2 when $M = 100$, $K = 4$, SNR $= -25$ dB and $\Delta_\theta = 10°$.

## VI. Conclusions

This paper addressed the problem of source enumeration in the non-white noises and under conditions of low sample support. The method applies a subspace averaging (SA) technique for increasing dimensions of the extracted subspaces. For each dimension, we get an estimate of the number of sources, and the final estimate is obtained by a majority vote rule. The method performs robustly under different noise models ranging from uncorrelated noises with different variances to arbitrarily correlated noises.

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