Locally Most Powerful Invariant Tests for the Properness of Quaternion Gaussian Vectors

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***Abstract*—Previous works have addressed the second-order statistical characterization of quaternion random vectors, introducing different properness definitions, and presenting the generalized likelihood ratio tests (GLRTs) for determining the kind of quaternion properness. This paper considers the more challenging problem of deriving the locally most powerful invariant tests (LMPITs), which can be obtained, even without an explicit expression for the maximal invariants, thanks to the Wijsman's theorem. Specifically, we consider three binary hypothesis testing problems involving the two main kinds of quaternion properness, and show that the LMPIT statistics are given by the Frobenius norm of three previously defined sample coherence matrices. The proposed detectors exhibit interesting connections with the problem of testing for the properness of a complex vector, and with the problems of testing for the sphericity of a four-dimensional real (or two-dimensional complex proper) vector. Additionally, some numerical examples show that in general, the proposed LMPITs outperform their GLRT counterparts, and in some cases the performance gap is very noticeable.**

***Index Terms*—Quaternions, properness, second-order circularity, locally most powerful invariant test (LMPIT), maximal invariant, Wijsman's theorem.**

I. Introduction

In the last decade, quaternion signal processing has attracted increasing attention due to its applications in image processing [1]–[5], computer graphics [6], aerospace and satellite tracking [7], [8], design and processing of space-time block codes [9]–[15], detection and processing of polarized waves [16]–[19], or modeling of wind profiles [20]–[23]. The application oriented efforts have been also complemented by some theoretical works, including those devoted to the statistical characterization of quaternion random vectors [24] (see also [25]–[27]).

The second-order statistical analysis of quaternion random vectors can be seen as a non-trivial extension of several previous results in the complex case [28]–[38]. In particular, a complex random vector is said to be proper if it is uncorrelated with its complex conjugate, which results in the optimality of the *conventional* linear processing. However, in the more general case of (possibly) improper complex vectors, the optimal linear processing is widely-linear, i.e., we have to simultaneously operate on the data vector and its complex conjugate. The selection of the most convenient type of processing is an important problem due to the fact that algorithms adapted for improper signals can fail or suffer from slow convergence when they are used for proper signals [39]. Thus, we should follow the principle of parsimony and choose the simplest model exploiting the statistical properties of the data, which requires to solve the problem of determining whether the complex data are proper or not [38], [40]–[46].

The quaternion case is more involved because there exist two main kinds of properness ($\mathbb{C}^\eta$-properness and $\mathbb{Q}$-properness), which also have direct implications on the structure of the optimal linear processing [24]. Specifically, the optimal linear processing of a quaternion random vector is in general *full-widely* linear, which requires the simultaneous operation on the quaternion vector and its involutions over three orthogonal pure quaternions $\{\eta, \eta', \eta''\}$. However, in the $\mathbb{C}^\eta$-proper case, we only need to operate on the quaternion vector and its involution over η, which is referred to as *semi-widely* linear processing, whereas in the $\mathbb{Q}$-proper case the *conventional* linear processing is optimal, i.e., we do not need to operate on the quaternion involutions. Therefore, due to the existence of three different kinds of linear processing, and analogously to the complex case, it becomes crucial to determine the kind of properness of a quaternion random vector [19]. Additionally, other potential applications of the quaternion properness tests include i) finding statistical invariances to rotations in imaging problems, ii) the statistical analysis of the dependencies among different trivariate signals, which could be related by means of random quaternions (representing rotations), and iii) the statistical characterization of the analytic signal extensions for trivariate vectors or bidimensional signals, which could benefit from a quaternionic representation resulting, in analogy with the unidimensional case, in proper analytic signals.

Focusing on the two main kinds of quaternion properness, we define three binary hypothesis testing problems, which have been previously approached by means of the corresponding generalized likelihood ratio tests (GLRTs). In particular, the three GLRTs were first proposed in [19], and the exact distribution of the test statistics, as well as several practical approximations, were presented in [47]. However, it is well known that the GLRT is not optimal in the Neyman-Pearson sense, and its performance can be seriously degraded for small sample sizes. This paper presents the locally most powerful invariant tests (LMPITs) for the three testing problems under the assumption of i.i.d. Gaussian data. That is, following the principle of invariance, and assuming that the two hypotheses

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are *very close*, we derive the best (in the Neyman-Pearson sense) invariant tests.

The principle of invariance is one of the fundamental ideas in hypothesis testing [48]–[51], and the traditional approach to derive optimal invariant tests consists in the identification of a maximal invariant [48], the derivation of its probability density function (pdf) under the two hypotheses, and the application of the Neyman-Pearson criterion. However, the theoretical derivation of the pdf's can be a very complicated task. Even worse, in some cases it is difficult to identify a maximal invariant statistic. Fortunately, this problem can be solved by means of the Wijsman's theorem [41], [52]–[54], sometimes also referred to as the Stein's theorem [51], [55], which states that the density ratio of the maximal invariant can be obtained by integrating over the group of transformations describing the problem invariances.

Although the identification of the maximal invariants is relatively easy in two out of the three binary hypothesis testing problems considered in this paper, the derivation of the LMPITs require the use of the Wijsman's theorem. Specifically, it is shown that the LMPIT statistics are given by the Frobenius norm of three previously defined sample coherence matrices [19], [24]. In comparison with the respective GLRTs, which are based on the determinants of the coherence matrices, the LMPITs require a lower number of vector observations (sample size). Moreover, it will be shown that the uniformly most powerful invariant test (UMPIT) only exists in a very particular situation, and we will explore the interesting connections with the problems of testing for the sphericity of a four-dimensional real (or two-dimensional complex proper) vector [56]–[58], as well as with the problem of testing for the properness of a complex random vector [40]–[43].

The paper is structured as follows: Section II introduces the basic concepts on quaternion algebra and summarizes the second-order statistical characterization of quaternion random vectors. Section III presents the three testing problems and briefly revisits some previous results. The main contributions of the paper are presented in Section IV, which addresses the problem of identifying the maximal invariants, presents the LMPITs, and analyzes their basic properties. The formal derivation of the LMPITs is relegated to Section V, and their practical performance is illustrated in Section VI by means of some numerical examples, which allow us to conclude that in general, the LMPITs outperform their GLRT counterparts. Finally, the paper conclusions are summarized in Section VII.

II. Preliminaries

In this paper we use bold-faced upper case letters to denote matrices, bold-faced lower case letters for column vectors, and light-faced lower case letters for scalar quantities. Superscripts $(\cdot)^*$, $(\cdot)^T$ and $(\cdot)^H$ denote quaternion (or complex) conjugate, transpose and Hermitian (i.e., transpose and quaternion conjugate), respectively. The notation $\mathbf{A} \in \mathbb{F}^{m\times n}$ denotes that $\mathbf{A}$ is a $m \times n$ matrix with entries in $\mathbb{F}$, where $\mathbb{F}$ can be $\mathbb{R}$, the field of real numbers, $\mathbb{C}$, the field of complex numbers, or $\mathbb{H}$, the skew-field of quaternion numbers. $\Re(\mathbf{A})$, $\text{Tr}(\mathbf{A})$ and $|\mathbf{A}|$ denote the real part, trace, and determinant of matrix $\mathbf{A}$. $\mathbf{A}^{1/2}$ (respectively $\mathbf{A}^{-1/2}$) is the Hermitian square root of the Hermitian matrix $\mathbf{A}$ (resp. $\mathbf{A}^{-1}$). The diagonal matrix with vector $\mathbf{a}$ along its diagonal is denoted by $\text{diag}(\mathbf{a})$, $\mathbf{I}_n$ is the identity matrix of dimension n, and $\mathbf{0}_{m\times n}$ is the $m \times n$ zero matrix. Finally, the Kronecker product is denoted by $\otimes$, E is the expectation operator, and in general $\mathbf{R}_{\mathbf{a},\mathbf{b}}$ is the cross-correlation matrix for vectors $\mathbf{a}$ and $\mathbf{b}$, i.e., $\mathbf{R}_{\mathbf{a},\mathbf{b}} = E\mathbf{a}\mathbf{b}^H$.

A. Quaternion Algebra

Quaternions are hypercomplex numbers defined by

$$x = r_1 + \eta r_\eta + \eta' r_{\eta'} + \eta'' r_{\eta''}, \tag{1}$$

where $r_1, r_\eta, r_{\eta'}, r_{\eta''} \in \mathbb{R}$ are four real numbers, and the three imaginary units[1] (η, η', η'') satisfy

$$\eta^2 = \eta'^2 = \eta''^2 = \eta\eta'\eta'' = -1, \tag{2}$$

which also implies $\eta\eta' = \eta''$, $\eta'\eta'' = \eta$, and $\eta''\eta = \eta'$.

Quaternions form a skew field $\mathbb{H}$ [59], which means that they satisfy the axioms of a field except the commutative law of the product, i.e., for $x, y \in \mathbb{H}$, $xy \neq yx$ in general. The conjugate of a quaternion x is $x^* = r_1 - \eta r_\eta - \eta' r_{\eta'} - \eta'' r_{\eta''}$, and the inner product of two quaternions x, y is defined as xy^*. Two quaternions are orthogonal if and only if (iff) their scalar product (the real part of the inner product) is zero, and the norm of a quaternion x is $|x| = \sqrt{xx^*} = \sqrt{r_1^2 + r_\eta^2 + r_{\eta'}^2 + r_{\eta''}^2}$. Furthermore, we say that ν is a pure unit quaternion iff $\nu^2 = -1$ (i.e., iff $|\nu| = 1$ and its real part is zero).

Quaternions admit the Euler representation

$$x = |x|e^{\nu\theta} = |x|\left(\cos\theta + \nu\sin\theta\right), \tag{3}$$

where ν is a pure unit quaternion and $\theta \in \mathbb{R}$ is the angle (or argument) of the quaternion. Taking this into account, we can easily define the rotation and involution operations [59]:

Definition 1 (Rotation and Involution): Consider a non-zero quaternion $a = |a|e^{\nu\theta} = |a|\left(\cos\theta + \nu\sin\theta\right)$, then

$$x^{(a)} = axa^{-1}, \tag{4}$$

represents a three-dimensional rotation of the imaginary part of x. Specifically, the vector $[r_\eta, r_{\eta'}, r_{\eta''}]^T$ is rotated an angle 2θ in the pure imaginary plane orthogonal to ν. In the particular case of pure quaternions ν, $x^{(\nu)}$ represents a rotation of angle π, which is an involution.

Finally, a quaternion x can also be represented by means of the *Cayley-Dickson* construction $x = a_1 + \eta'' a_2$, where

$$a_1 = r_1 + \eta r_\eta, \qquad a_2 = r_{\eta''} + \eta r_{\eta'}, \tag{5}$$

can be seen as complex numbers in the plane $\{1, \eta\}$.

[1] In this paper we use the general representation $\{\eta, \eta', \eta''\}$ instead of the conventional canonical basis $\{i, j, k\}$.

B. Second-Order Statistics of Quaternion Random Vectors

The second-order statistics (SOS) of a n-dimensional quaternion random vector $\mathbf{x} = \mathbf{r}_1 + \eta\mathbf{r}_\eta + \eta'\mathbf{r}_{\eta'} + \eta''\mathbf{r}_{\eta''}$ are obviously given by the joint SOS of the vectors $\mathbf{r}_1, \mathbf{r}_\eta, \mathbf{r}_{\eta'}, \mathbf{r}_{\eta''} \in \mathbb{R}^{n\times 1}$ in its real representation. However, analogously to the case of complex random vectors [28]–[33], [35], [37], the statistical analysis can benefit from the definition of an augmented quaternion vector[2]

$$\bar{\mathbf{x}} = \begin{bmatrix} \mathbf{x} \\ \mathbf{x}^{(\eta)} \\ \mathbf{x}^{(\eta')} \\ \mathbf{x}^{(\eta'')} \end{bmatrix} = 2\mathbf{T}_n\mathbf{r}, \tag{6}$$

where $\mathbf{r} = \left[\mathbf{r}_1^T, \mathbf{r}_\eta^T, \mathbf{r}_{\eta'}^T, \mathbf{r}_{\eta''}^T\right]^T$, and $\mathbf{T}_n \in \mathbb{H}^{4n\times 4n}$ is a unitary matrix defined as

$$\mathbf{T}_n = \frac{1}{2}\begin{bmatrix} +1 & +\eta & +\eta' & +\eta'' \\ +1 & +\eta & -\eta' & -\eta'' \\ +1 & -\eta & +\eta' & -\eta'' \\ +1 & -\eta & -\eta' & +\eta'' \end{bmatrix} \otimes \mathbf{I}_n. \tag{7}$$

Thus, the SOS of $\mathbf{x}$ are given by the augmented covariance matrix [24]

$$\begin{aligned}\mathbf{R}_{\bar{\mathbf{x}},\bar{\mathbf{x}}} &= 4\mathbf{T}_n\mathbf{R}_{\mathbf{r},\mathbf{r}}\mathbf{T}_n^H \\ &= \begin{bmatrix} \mathbf{R}_{\mathbf{x},\mathbf{x}} & \mathbf{R}_{\mathbf{x},\mathbf{x}^{(\eta)}} & \mathbf{R}_{\mathbf{x},\mathbf{x}^{(\eta')}} & \mathbf{R}_{\mathbf{x},\mathbf{x}^{(\eta'')}} \\ \mathbf{R}^{(\eta)}_{\mathbf{x},\mathbf{x}^{(\eta)}} & \mathbf{R}^{(\eta)}_{\mathbf{x},\mathbf{x}} & \mathbf{R}^{(\eta)}_{\mathbf{x},\mathbf{x}^{(\eta'')}} & \mathbf{R}^{(\eta)}_{\mathbf{x},\mathbf{x}^{(\eta')}} \\ \mathbf{R}^{(\eta')}_{\mathbf{x},\mathbf{x}^{(\eta')}} & \mathbf{R}^{(\eta')}_{\mathbf{x},\mathbf{x}^{(\eta'')}} & \mathbf{R}^{(\eta')}_{\mathbf{x},\mathbf{x}} & \mathbf{R}^{(\eta')}_{\mathbf{x},\mathbf{x}^{(\eta)}} \\ \mathbf{R}^{(\eta'')}_{\mathbf{x},\mathbf{x}^{(\eta'')}} & \mathbf{R}^{(\eta'')}_{\mathbf{x},\mathbf{x}^{(\eta')}} & \mathbf{R}^{(\eta'')}_{\mathbf{x},\mathbf{x}^{(\eta)}} & \mathbf{R}^{(\eta'')}_{\mathbf{x},\mathbf{x}} \end{bmatrix}, \end{aligned} \tag{8}$$

which contains the covariance matrix $\mathbf{R}_{\mathbf{x},\mathbf{x}} = E\mathbf{x}\mathbf{x}^H$ and three complementary covariance matrices $\mathbf{R}_{\mathbf{x},\mathbf{x}^{(\eta)}} = E\mathbf{x}\mathbf{x}^{(\eta)^H}$, $\mathbf{R}_{\mathbf{x},\mathbf{x}^{(\eta')}} = E\mathbf{x}\mathbf{x}^{(\eta')^H}$ and $\mathbf{R}_{\mathbf{x},\mathbf{x}^{(\eta'')}} = E\mathbf{x}\mathbf{x}^{(\eta'')^H}$. Interestingly, this representation allows us to easily relate the SOS of the quaternion vector $\mathbf{x}$ and those of some common transformations [24]:

Lemma 1: Consider the *full-widely* linear transformation

$$\mathbf{y} = \mathbf{F}_{\bar{\mathbf{x}}}^H\bar{\mathbf{x}} = \mathbf{F}_1^H\mathbf{x} + \mathbf{F}_\eta^H\mathbf{x}^{(\eta)} + \mathbf{F}_{\eta'}^H\mathbf{x}^{(\eta')} + \mathbf{F}_{\eta''}^H\mathbf{x}^{(\eta'')}, \tag{9}$$

where $\mathbf{F}_{\bar{\mathbf{x}}} = \left[\mathbf{F}_1^T, \mathbf{F}_\eta^T, \mathbf{F}_{\eta'}^T, \mathbf{F}_{\eta''}^T\right]^T \in \mathbb{H}^{4n\times n}$. Then, the SOS of $\mathbf{y}$ are given by $\mathbf{R}_{\bar{\mathbf{y}},\bar{\mathbf{y}}} = \overline{\mathbf{F}}^H\mathbf{R}_{\bar{\mathbf{x}},\bar{\mathbf{x}}}\overline{\mathbf{F}}$, where

$$\overline{\mathbf{F}} = \underbrace{\begin{bmatrix} \mathbf{F}_1 & \mathbf{F}_\eta^{(\eta)} & \mathbf{F}_{\eta'}^{(\eta')} & \mathbf{F}_{\eta''}^{(\eta'')} \\ \mathbf{F}_\eta & \mathbf{F}_1^{(\eta)} & \mathbf{F}_{\eta''}^{(\eta')} & \mathbf{F}_{\eta'}^{(\eta'')} \\ \mathbf{F}_{\eta'} & \mathbf{F}_{\eta''}^{(\eta)} & \mathbf{F}_1^{(\eta')} & \mathbf{F}_\eta^{(\eta'')} \\ \mathbf{F}_{\eta''} & \mathbf{F}_{\eta'}^{(\eta)} & \mathbf{F}_\eta^{(\eta')} & \mathbf{F}_1^{(\eta'')} \end{bmatrix}}_{4n\times 4n}. \tag{10}$$

Lemma 2: A rotation $\mathbf{y} = \mathbf{x}^{(a)}$ results in a simultaneous rotation of the orthogonal basis $\{1, \eta, \eta', \eta''\}$ and the augmented covariance matrix

$$\mathbf{R}_{\bar{\mathbf{y}},\bar{\mathbf{y}}}(\{1, \eta, \eta', \eta''\}) = \mathbf{R}^{(a)}_{\bar{\mathbf{x}},\bar{\mathbf{x}}}(\{1, \eta^{(a^*)}, \eta'^{(a^*)}, \eta''^{(a^*)}\}), \tag{11}$$

where the expressions in parentheses make explicit the bases for the augmented covariance matrices.

Lemma 3: The augmented covariance matrices in two different orthogonal bases are related as

$$\mathbf{R}_{\bar{\mathbf{x}},\bar{\mathbf{x}}}(\{1, \nu, \nu', \nu''\}) = \mathbf{\Gamma}\mathbf{R}_{\bar{\mathbf{x}},\bar{\mathbf{x}}}(\{1, \eta, \eta', \eta''\})\mathbf{\Gamma}^H, \tag{12}$$

where

$$\mathbf{\Gamma} = \begin{bmatrix} 1 & \mathbf{0}_{1\times 3} \\ \mathbf{0}_{3\times 1} & \mathbf{\Lambda}_\nu\mathbf{Q}\mathbf{\Lambda}_\eta^H \end{bmatrix} \otimes \mathbf{I}_n, \tag{13}$$

$\mathbf{Q} \in \mathbb{R}^{3\times 3}$ is the rotation matrix for the change of basis $[\nu, \nu', \nu''] = [\eta, \eta', \eta'']\mathbf{Q}^T$, $\mathbf{\Lambda}_\nu = \operatorname{diag}\left([\nu, \nu', \nu'']^T\right)$, and $\mathbf{\Lambda}_\eta = \operatorname{diag}\left([\eta, \eta', \eta'']^T\right)$.

Finally, we conclude this subsection by introducing a useful factorization for complementary covariance matrices and, in general, for η-Hermitian quaternion matrices [60], i.e. matrices $\mathbf{A} \in \mathbb{H}^{n\times n}$ satisfying $\mathbf{A}^H = \mathbf{A}^{(\eta)}$. This tool can be seen as a quaternion extension of the Takagi's factorization for symmetric complex matrices [61], and it is easy to check that both factorizations coincide when $\mathbf{A}$ is a *complex* matrix with its imaginary part in the plane $\{\eta', \eta''\}$. For completeness, we include a proof which avoids the problems encountered in the case of singular values with multiplicities [60].

Lemma 4 (Unitary Factorization of η-Hermitian Matrices): Every η-Hermitian matrix $\mathbf{A} \in \mathbb{H}^{n\times n}$ admits a factorization $\mathbf{A} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^{(\eta)^H}$, where $\mathbf{U} \in \mathbb{H}^{n\times n}$ is a unitary matrix with the singular vectors, and $\mathbf{\Lambda} \in \mathbb{R}^{n\times n}$ is a diagonal matrix with the singular values.

Proof: Let us start by writing the Cayley-Dickson representation $\mathbf{A} = \mathbf{A}_1 + \eta\mathbf{A}_2$, where $\mathbf{A}_1, \mathbf{A}_2 \in \mathbb{H}^{n\times n}$ belong to the plane $\{1, \eta'\}$, i.e., they can be seen as complex matrices. Thus, it is easy to verify that the η-Hermitian condition $\mathbf{A}^H = \mathbf{A}^{(\eta)}$ implies $\mathbf{A}_1 = \mathbf{A}_1^T$ and $\mathbf{A}_2 = -\mathbf{A}_2^H$. Now, defining the adjoint matrix

$$\breve{\mathbf{A}} = \begin{bmatrix} \mathbf{A}_1 & -\mathbf{A}_2^* \\ \mathbf{A}_2 & \mathbf{A}_1^* \end{bmatrix}, \tag{14}$$

and using the conventional Takagi's [61] factorization for complex symmetric matrices, we can obtain the SVD-like decomposition $\breve{\mathbf{A}} = \breve{\mathbf{U}}\breve{\mathbf{\Lambda}}\breve{\mathbf{U}}^T$, where the unitary matrix $\breve{\mathbf{U}}$ belongs to the plane $\{1, \eta'\}$. Moreover, from the uniqueness of the Takagi's factorization, we can conclude that the matrices $\breve{\mathbf{U}}$ and $\breve{\mathbf{\Lambda}}$ have the structure

$$\breve{\mathbf{U}} = \begin{bmatrix} \mathbf{U}_1 & -\mathbf{U}_2^* \\ \mathbf{U}_2 & \mathbf{U}_1^* \end{bmatrix}, \qquad \breve{\mathbf{\Lambda}} = \begin{bmatrix} \mathbf{\Lambda} & \mathbf{0}_{n\times n} \\ \mathbf{0}_{n\times n} & \mathbf{\Lambda} \end{bmatrix}, \tag{15}$$

with $\mathbf{U}_1, \mathbf{U}_2 \in \mathbb{H}^{n\times n}$. Finally, using the Cayley-Dickson construction $\mathbf{U} = \mathbf{U}_1 + \eta\mathbf{U}_2$, the decomposition $\breve{\mathbf{A}} = \breve{\mathbf{U}}\breve{\mathbf{\Lambda}}\breve{\mathbf{U}}^T$ can be compactly written as $\mathbf{A} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^{(\eta)^H}$. ■

C. Properness of Quaternion Random Vectors

Analogously to the complex case [37], the structure of the optimal linear processing of quaternion random vectors depends on the quaternion properness. In [24] (see also [25]–[27]), the authors have presented two main kinds of quaternion properness:

[2] From now on, we will use the notation $\mathbf{F}^{(a)}$ to denote the element-wise rotation of matrix $\mathbf{F}$.

Definition 2 ($\mathbb{Q}$-Properness): A quaternion random vector $\mathbf{x}$ is $\mathbb{Q}$-proper iff the three complementary covariance matrices $\mathbf{R}_{\mathbf{x},\mathbf{x}^{(\eta)}}$, $\mathbf{R}_{\mathbf{x},\mathbf{x}^{(\eta')}}$ and $\mathbf{R}_{\mathbf{x},\mathbf{x}^{(\eta'')}}$ vanish.

Definition 3 ($\mathbb{C}^\eta$-Properness): A quaternion random vector $\mathbf{x}$ is $\mathbb{C}^\eta$-proper iff the complementary covariance matrices $\mathbf{R}_{\mathbf{x},\mathbf{x}^{(\eta')}}$ and $\mathbf{R}_{\mathbf{x},\mathbf{x}^{(\eta'')}}$ vanish.

As a direct consequence of Lemma 3, $\mathbb{Q}$-properness implies $\mathbb{C}^\eta$-properness for all pure quaternions η. Furthermore, the $\mathbb{C}^\eta$-properness definition is directly related to the complex properness of the vectors in the Cayley-Dickson representation of $\mathbf{x}$ [24]:

Lemma 5: A quaternion random vector $\mathbf{x}$ is $\mathbb{C}^\eta$-proper iff the complex vectors $\mathbf{a}_1, \mathbf{a}_2 \in \mathbb{C}^{n\times 1}$ in its Cayley-Dickson representation $\mathbf{x} = \mathbf{a}_1 + \eta''\mathbf{a}_2$ are jointly proper, i.e., iff the complex vector $\mathbf{a} = [\mathbf{a}_1^T, \mathbf{a}_2^T]^T$ is proper ($\mathbf{R}_{\mathbf{a},\mathbf{a}^*} = \mathbf{0}_{2n\times 2n}$).

From a practical point of view, the main implications of the properness definitions consist in the simplification of the optimal linear processing of quaternion random vectors. In the general case, the optimal linear processing is *full-widely* linear, i.e., we must simultaneously operate on the quaternion random vector and its three involutions. However, in the case of proper vectors the optimal linear processing simplifies as follows [24]:

Lemma 6 (Semi-widely linear processing): The optimal linear processing of $\mathbb{C}^\eta$-proper vectors is semi-widely linear

$$\mathbf{y} = \mathbf{F}_1^H\mathbf{x} + \mathbf{F}_\eta^H\mathbf{x}^{(\eta)}. \tag{16}$$

Lemma 7 (Conventional linear processing): The optimal linear processing of $\mathbb{Q}$-proper vectors takes the form

$$\mathbf{y} = \mathbf{F}_1^H\mathbf{x}, \tag{17}$$

i.e., we do not need to operate on the quaternion involutions.

Finally, in [24] the authors introduced a third kind of quaternion properness, which can be interpreted as the *difference* between $\mathbb{C}^\eta$ and $\mathbb{Q}$ properness.

Definition 4 ($\mathbb{R}^\eta$-Properness): A quaternion random vector $\mathbf{x}$ is $\mathbb{R}^\eta$-proper iff the complementary covariance matrix $\mathbf{R}_{\mathbf{x},\mathbf{x}^{(\eta)}}$ vanishes.

III. Testing for Quaternion Properness: Problem Statement and Previous Results

A. Problem Formulation

Analogously to the complex case [42], [43], determining the kind of properness of a quaternion random vector is an important problem because it establishes the most convenient kind of linear processing. Here, we consider the three following hypotheses:

- $\mathcal{H}_\mathbb{Q}$: The quaternion random vector $\mathbf{x}$ is $\mathbb{Q}$-proper.
- $\mathcal{H}_{\mathbb{C}^\eta}$: The quaternion random vector $\mathbf{x}$ is $\mathbb{C}^\eta$-proper.
- $\mathcal{H}_\mathcal{I}$: The quaternion random vector $\mathbf{x}$ is not constrained to be $\mathbb{Q}$-proper nor $\mathbb{C}^\eta$-proper.

Depending on the particular problem/application, we could have some a priori information. For instance, if we know that the quaternion random vector has been obtained from two jointly-proper complex vectors $\mathbf{a}_1$, $\mathbf{a}_2$ in the plane $\{1,\eta\}$, we should take into account that $\mathbf{x} = \mathbf{a}_1 + \eta''\mathbf{a}_2$ is $\mathbb{C}^\eta$-proper. Thus, in this paper we consider three different binary hypothesis testing problems:

- $\mathbb{Q}$-properness test: This is the problem of determining whether $\mathbf{x}$ is $\mathbb{Q}$-proper or not. That is, we are testing the hypothesis $\mathcal{H}_\mathbb{Q}$ versus $\mathcal{H}_\mathcal{I}$.
- $\mathbb{C}^\eta$-properness test: The problem of determining whether $\mathbf{x}$ is $\mathbb{C}^\eta$-proper or not. In other words, this is the problem of testing $\mathcal{H}_{\mathbb{C}^\eta}$ versus $\mathcal{H}_\mathcal{I}$.
- $\mathbb{Q}$-properness versus $\mathbb{C}^\eta$-properness test: This is the problem of determining whether the $\mathbb{C}^\eta$-proper vector $\mathbf{x}$ is also $\mathbb{Q}$-proper, i.e., $\mathcal{H}_\mathbb{Q}$ versus $\mathcal{H}_{\mathbb{C}^\eta}$.

In order to solve these binary hypothesis testing problems, we will assume T i.i.d. realizations of a zero-mean quaternion Gaussian vector. Therefore, the tests to be presented do not need to be optimal in any sense for non-Gaussian distributions or non i.i.d. vector realizations.[3] Although the case of non-Gaussian data has been addressed in [47], where the authors proposed a modified version of the GLRT, the derivation of tests for more general families of distributions and for correlated data is an interesting topic for future research, but it is beyond the scope of this paper.

With the above assumptions, the probability density function (pdf) of the random vector $\mathbf{x} \in \mathbb{H}^{n\times 1}$ is [24]

$$p(\mathbf{x}) = \frac{1}{(\pi/2)^{2n}|\mathbf{R}_{\bar{\mathbf{x}},\bar{\mathbf{x}}}|^{1/2}} \exp\left(-\frac{1}{2}\bar{\mathbf{x}}^H\mathbf{R}_{\bar{\mathbf{x}},\bar{\mathbf{x}}}^{-1}\bar{\mathbf{x}}\right), \tag{18}$$

and from the T i.i.d. realizations $\mathbf{x}[t]$ ($t = 0,\ldots,T-1$) we can define the augmented sample-covariance matrix

$$\begin{aligned}\hat{\mathbf{R}}_{\bar{\mathbf{x}},\bar{\mathbf{x}}} &= \frac{1}{T}\sum_{t=0}^{T-1}\bar{\mathbf{x}}[t]\bar{\mathbf{x}}^H[t] \\ &= \begin{bmatrix} \hat{\mathbf{R}}_{\mathbf{x},\mathbf{x}} & \hat{\mathbf{R}}_{\mathbf{x},\mathbf{x}^{(\eta)}} & \hat{\mathbf{R}}_{\mathbf{x},\mathbf{x}^{(\eta')}} & \hat{\mathbf{R}}_{\mathbf{x},\mathbf{x}^{(\eta'')}} \\ \hat{\mathbf{R}}^{(\eta)}_{\mathbf{x},\mathbf{x}^{(\eta)}} & \hat{\mathbf{R}}^{(\eta)}_{\mathbf{x},\mathbf{x}} & \hat{\mathbf{R}}_{\mathbf{x},\mathbf{x}^{(\eta'')}} & \hat{\mathbf{R}}_{\mathbf{x},\mathbf{x}^{(\eta')}} \\ \hat{\mathbf{R}}^{(\eta')}_{\mathbf{x},\mathbf{x}^{(\eta')}} & \hat{\mathbf{R}}^{(\eta')}_{\mathbf{x},\mathbf{x}^{(\eta'')}} & \hat{\mathbf{R}}^{(\eta')}_{\mathbf{x},\mathbf{x}} & \hat{\mathbf{R}}^{(\eta')}_{\mathbf{x},\mathbf{x}^{(\eta')}} \\ \hat{\mathbf{R}}^{(\eta'')}_{\mathbf{x},\mathbf{x}^{(\eta'')}} & \hat{\mathbf{R}}^{(\eta'')}_{\mathbf{x},\mathbf{x}^{(\eta')}} & \hat{\mathbf{R}}^{(\eta'')}_{\mathbf{x},\mathbf{x}^{(\eta)}} & \hat{\mathbf{R}}^{(\eta'')}_{\mathbf{x},\mathbf{x}} \end{bmatrix},\end{aligned} \tag{19}$$

which also provides obvious definitions of the sample covariance $\hat{\mathbf{R}}_{\mathbf{x},\mathbf{x}}$ and complementary covariance $\hat{\mathbf{R}}_{\mathbf{x},\mathbf{x}^{(\eta)}}$, $\hat{\mathbf{R}}_{\mathbf{x},\mathbf{x}^{(\eta')}}$, $\hat{\mathbf{R}}_{\mathbf{x},\mathbf{x}^{(\eta'')}}$ matrices.

B. Previous Works: Generalized Likelihood Ratio Tests

The three proposed binary hypothesis testing problems have been previously considered in [19], [47]. In particular, in [19] the authors presented the three associated generalized likelihood ratio tests (GLRTs), as well as the asymptotic distribution of the test statistics and a multiple hypotheses testing procedure based on the combination of the three GLRTs. In [47], the authors derived the three GLRTs in an alternative way, and provided the exact distribution of the test

[3]Nevertheless, the proposed tests could be easily adapted to some relatively simple cases, such as the signal model $\begin{bmatrix}\mathbf{x}[0] & \cdots & \mathbf{x}[T-1]\end{bmatrix} = \begin{bmatrix}\mathbf{s}[0] & \cdots & \mathbf{s}[T-1]\end{bmatrix}\mathbf{B}$, where $\mathbf{x}[t]$ ($t = 0,\ldots,T-1$) denotes the correlated observations, $\mathbf{s}[t]$ represents T i.i.d. realizations of a quaternion Gaussian vector $\mathbf{s}$, and $\mathbf{B} \in \mathbb{H}^{T\times T}$ controls the *horizontal* correlation. In this case, and assuming that the matrix $\mathbf{B}$ is known, the observations $\mathbf{x}[t]$ can be *horizontally* prewhitened, recovering the model addressed in this paper.

statistics, as well as several practical approximations. In this subsection, we summarize the main results in [19], which also allows us to introduce some definitions that will be useful in the following sections. In particular, we define the block-diagonal matrices

$$\hat{\mathbf{D}}_{\mathbb{C}^\eta} = \begin{bmatrix} \hat{\mathbf{R}}_{\mathbf{x},\mathbf{x}} & \hat{\mathbf{R}}_{\mathbf{x},\mathbf{x}^{(\eta)}} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} \\ \hat{\mathbf{R}}^{(\eta)}_{\mathbf{x},\mathbf{x}^{(\eta)}} & \hat{\mathbf{R}}^{(\eta)}_{\mathbf{x},\mathbf{x}} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} \\ \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & \hat{\mathbf{R}}^{(\eta')}_{\mathbf{x},\mathbf{x}} & \hat{\mathbf{R}}^{(\eta')}_{\mathbf{x},\mathbf{x}^{(\eta')}} \\ \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & \hat{\mathbf{R}}^{(\eta'')}_{\mathbf{x},\mathbf{x}^{(\eta)}} & \hat{\mathbf{R}}^{(\eta'')}_{\mathbf{x},\mathbf{x}} \end{bmatrix}, \quad (20)$$

and

$$\hat{\mathbf{D}}_{\mathbb{Q}} = \begin{bmatrix} \hat{\mathbf{R}}_{\mathbf{x},\mathbf{x}} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} \\ \mathbf{0}_{n\times n} & \hat{\mathbf{R}}^{(\eta)}_{\mathbf{x},\mathbf{x}} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} \\ \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & \hat{\mathbf{R}}^{(\eta')}_{\mathbf{x},\mathbf{x}} & \mathbf{0}_{n\times n} \\ \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & \hat{\mathbf{R}}^{(\eta'')}_{\mathbf{x},\mathbf{x}} \end{bmatrix}, \quad (21)$$

and summarize the main results in [19] in Table I, which shows the three GLRT statistics. The GLRTs reject the null (proper) hypothesis for high values of $\hat{\mathcal{P}}_{\mathbb{Q}}$, $\hat{\mathcal{P}}_{\mathbb{C}^\eta}$ or $\hat{\mathcal{P}}_{\mathbb{R}^\eta}$, which can be seen as estimates of the improperness measures presented in [24]. Specifically, the test statistics are obtained from the sample coherence matrices

$$\hat{\mathbf{\Phi}}_{\mathbb{Q}} = \hat{\mathbf{D}}_{\mathbb{Q}}^{-1/2}\hat{\mathbf{R}}_{\bar{\mathbf{x}},\bar{\mathbf{x}}}\hat{\mathbf{D}}_{\mathbb{Q}}^{-1/2}, \quad (22)$$

$$\hat{\mathbf{\Phi}}_{\mathbb{C}^\eta} = \hat{\mathbf{D}}_{\mathbb{C}^\eta}^{-1/2}\hat{\mathbf{R}}_{\bar{\mathbf{x}},\bar{\mathbf{x}}}\hat{\mathbf{D}}_{\mathbb{C}^\eta}^{-1/2}, \quad (23)$$

$$\hat{\mathbf{\Phi}}_{\mathbb{R}^\eta} = \hat{\mathbf{D}}_{\mathbb{Q}}^{-1/2}\hat{\mathbf{D}}_{\mathbb{C}^\eta}\hat{\mathbf{D}}_{\mathbb{Q}}^{-1/2}, \quad (24)$$

and they satisfy the relationship

$$\hat{\mathcal{P}}_{\mathbb{Q}} = \hat{\mathcal{P}}_{\mathbb{C}^\eta} + \hat{\mathcal{P}}_{\mathbb{R}^\eta}, \quad (25)$$

which has been used in [19] for introducing a multiple hypotheses test based on the three previous measures. Moreover, using the Cayley-Dickson representation, we can rewrite the augmented quaternion vector as $\bar{\mathbf{x}} = \left[\tilde{\mathbf{x}}^T, \tilde{\mathbf{x}}^{(\eta')^T}\right]^T$, where the semi-augmented quaternion vector $\tilde{\mathbf{x}}$ is given by [24]

$$\underbrace{\begin{bmatrix} \mathbf{x} \\ \mathbf{x}^{(\eta)} \end{bmatrix}}_{\tilde{\mathbf{x}}} = \sqrt{2}\underbrace{\left(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & \eta'' \\ 1 & -\eta'' \end{bmatrix} \otimes \mathbf{I}_n\right)}_{\mathbf{L}}\underbrace{\begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{bmatrix}}_{\mathbf{a}}. \quad (26)$$

Thus, taking into account the unitarity of the operator $\mathbf{L}$, it is easy to prove that the sample $\mathbb{C}^\eta$-coherence matrix can be rewritten as

$$\hat{\mathbf{\Phi}}_{\mathbb{C}^\eta} = \begin{bmatrix} \mathbf{L} & \mathbf{0}_{2n\times 2n} \\ \mathbf{0}_{2n\times 2n} & \mathbf{L}^{(\eta')} \end{bmatrix} \hat{\mathbf{\Phi}}_{\tilde{\mathbf{a}}} \begin{bmatrix} \mathbf{L} & \mathbf{0}_{2n\times 2n} \\ \mathbf{0}_{2n\times 2n} & \mathbf{L}^{(\eta')} \end{bmatrix}^H, \quad (27)$$

where $\hat{\mathbf{\Phi}}_{\tilde{\mathbf{a}}} = \hat{\mathbf{D}}_{\tilde{\mathbf{a}}}^{-\frac{1}{2}}\hat{\mathbf{R}}_{\tilde{\mathbf{a}},\tilde{\mathbf{a}}}\hat{\mathbf{D}}_{\tilde{\mathbf{a}}}^{-\frac{H}{2}}$ is the sample coherence matrix for the complex vector $\tilde{\mathbf{a}} = \left[\mathbf{a}^T, \mathbf{a}^H\right]^T$, and

$$\hat{\mathbf{D}}_{\tilde{\mathbf{a}}} = \begin{bmatrix} \hat{\mathbf{R}}_{\mathbf{a},\mathbf{a}} & \mathbf{0}_{2n\times 2n} \\ \mathbf{0}_{2n\times 2n} & \hat{\mathbf{R}}^*_{\mathbf{a},\mathbf{a}} \end{bmatrix}. \quad (28)$$

Therefore, we can conclude that the $\mathbb{C}^\eta$-properness GLRT is equivalent to the GLRT for testing the properness of the complex vector $\mathbf{a} = [\mathbf{a}_1^T, \mathbf{a}_2^T]^T$, or equivalently, for determining whether $\mathbf{a}_1$ and $\mathbf{a}_2$ are jointly complex-proper or not [19], [24], [38], [42]–[46].

TABLE I
GLRT STATISTICS

Test	Sample Size	GLRT statistic
$\mathcal{H}_{\mathbb{Q}}$ vs. $\mathcal{H}_{\mathcal{I}}$	$T \geq 4n$	$\hat{\mathcal{P}}_{\mathbb{Q}} = -\frac{1}{2}\ln\left\|\hat{\mathbf{\Phi}}_{\mathbb{Q}}\right\|$
$\mathcal{H}_{\mathbb{C}^\eta}$ vs. $\mathcal{H}_{\mathcal{I}}$	$T \geq 4n$	$\hat{\mathcal{P}}_{\mathbb{C}^\eta} = -\frac{1}{2}\ln\left\|\hat{\mathbf{\Phi}}_{\mathbb{C}^\eta}\right\|$
$\mathcal{H}_{\mathbb{Q}}$ vs. $\mathcal{H}_{\mathbb{C}^\eta}$	$T \geq 2n$	$\hat{\mathcal{P}}_{\mathbb{R}^\eta} = -\frac{1}{2}\ln\left\|\hat{\mathbf{\Phi}}_{\mathbb{R}^\eta}\right\|$

IV. Locally Most Powerful Invariant Tests

Although the GLRTs are simple detectors with nice detection rules, they can suffer from poor performance, especially for small sample sizes T. This motivates us to consider the derivation of the locally most powerful invariant tests (LMPITs), i.e., the most powerful tests (in the Neyman-Pearson sense) among those preserving the particular invariances [48], [49] of each testing problem, when the null (proper) and alternative hypotheses are *very close*. In this section, we start by summarizing the invariances of each testing problem, which is complemented with a discussion on the derivation of the maximal invariants. Finally, we introduce the LMPITs, analyze their properties, and point out the connections with the LMPITs for some related testing problems.

A. Problem Invariances and Maximal Invariants

Before proceeding, let us summarize the invariances of the two main quaternion properness definitions:

Property 1 ($\mathbb{Q}$-Properness Invariances): The $\mathbb{Q}$-properness definition is invariant to rotations and invertible *conventional* linear transformations, i.e., $\mathbf{x}$ is $\mathbb{Q}$-proper iff $\mathbf{y} = \mathbf{F}_1^H\mathbf{x}^{(a)}$ is $\mathbb{Q}$-proper for all non-null $a \in \mathbb{H}$ and invertible $\mathbf{F}_1 \in \mathbb{H}^{n\times n}$.

Property 2 ($\mathbb{C}^\eta$-Properness Invariances): The $\mathbb{C}^\eta$-properness definition is invariant to invertible *semi-widely* linear transformations, i.e., $\mathbf{x}$ is $\mathbb{C}^\eta$-proper iff $\mathbf{y} = \mathbf{F}_1^H\mathbf{x} + \mathbf{F}_\eta^H\mathbf{x}^{(\eta)}$ is $\mathbb{C}^\eta$-proper for all $\mathbf{F}_1, \mathbf{F}_\eta \in \mathbb{H}^{n\times n}$ resulting in an invertible transformation $\bar{\mathbf{y}} = \overline{\mathbf{F}}^H\bar{\mathbf{x}}$.

These properties, which follow directly from Lemmas 1-3 and the properness definitions, allow us to establish the invariances of the three binary hypothesis tests:

Lemma 8 (Invariances of the $\mathbb{Q}$-Properness test): The problem of testing $\mathcal{H}_{\mathbb{Q}}$ versus $\mathcal{H}_{\mathcal{I}}$ is invariant under the group $\mathcal{G}_{\mathbb{Q}}$ of quaternion rotations and invertible *conventional* linear transformations.

Lemma 9 (Invariances of the $\mathbb{C}^\eta$-Properness test): The problem of testing $\mathcal{H}_{\mathbb{C}^\eta}$ versus $\mathcal{H}_{\mathcal{I}}$ is invariant under the group $\mathcal{G}_{\mathbb{C}^\eta}$ of invertible *semi-widely* linear transformations.

Lemma 10 (Invariances of the $\mathcal{H}_{\mathbb{Q}}$ versus $\mathcal{H}_{\mathbb{C}^\eta}$ test): The problem of testing $\mathcal{H}_{\mathbb{Q}}$ versus $\mathcal{H}_{\mathbb{C}^\eta}$ is invariant under the group $\mathcal{G}_{\mathbb{R}^\eta}$ of invertible *conventional* linear transformations.

Here, we must point out that the three GLRTs presented in the previous section preserve the invariances of the corresponding testing problems. In other words, the three GLRTs belong to the class of invariant detectors to be explored in the rest of the paper. Finally, taking into account the test invariances, we are ready to consider the derivation of the associated maximal invariants [48], [49].

1) Maximal invariant for $\mathcal{H}_{\mathbb{Q}}$ versus $\mathcal{H}_{\mathbb{C}^\eta}$: It can be easily proved that the sufficient statistic for this testing problem is $\hat{\mathbf{D}}_{\mathbb{C}^\eta}$, and taking into account the invariance under invertible *conventional* linear transformations, we can introduce a transformation $\mathbf{y}[t] = \mathbf{F}_1^H \mathbf{x}[t]$ such that $\hat{\mathbf{R}}_{\mathbf{y},\mathbf{y}} = \mathbf{I}_n$ and $\hat{\mathbf{R}}_{\mathbf{y},\mathbf{y}^{(\eta)}}$ is a real diagonal matrix, where the entries in the diagonal are given by the (ordered) sample canonical correlations [24], [62] between the random vectors $\mathbf{x}$ and $\mathbf{x}^{(\eta)}$.[4] Thus, the n sample canonical correlations constitute a maximal invariant (under the group of invertible *conventional* linear transformations) for testing $\mathcal{H}_{\mathbb{Q}}$ versus $\mathcal{H}_{\mathbb{C}^\eta}$. Moreover, it is straightforward to prove that there exists a one-to-one correspondence between the n sample canonical correlations and the eigenvalues of the sample $\mathbb{R}^\eta$-coherence matrix $\hat{\mathbf{\Phi}}_{\mathbb{R}^\eta}$, i.e., we can consider the eigenvalues of $\hat{\mathbf{\Phi}}_{\mathbb{R}^\eta}$ as the maximal invariant.

2) Maximal invariant for the $\mathbb{C}^\eta$-Properness test: In this case the sufficient statistic is $\hat{\mathbf{R}}_{\bar{\mathbf{x}},\bar{\mathbf{x}}}$, and taking into account the invariance of the testing problem under invertible *semi-widely* linear transformations, we can introduce a transformation $\mathbf{y}[t] = \mathbf{F}_1^H \mathbf{x}[t] + \mathbf{F}_\eta^H \mathbf{x}^{(\eta)}[t]$ such that

$$\hat{\mathbf{R}}_{\mathbf{y},\mathbf{y}} = \mathbf{I}_n, \tag{29}$$

$$\hat{\mathbf{R}}_{\mathbf{y},\mathbf{y}^{(\eta)}} = \mathbf{0}_{n\times n}, \tag{30}$$

$$\hat{\mathbf{R}}_{\mathbf{y},\mathbf{y}^{(\eta')}} = \hat{\mathbf{\Sigma}}_{\eta'} = \text{diag}\left(\hat{\mathbf{c}}_1\right) - \text{diag}\left(\hat{\mathbf{c}}_2\right), \tag{31}$$

$$\hat{\mathbf{R}}_{\mathbf{y},\mathbf{y}^{(\eta'')}} = \hat{\mathbf{\Sigma}}_{\eta''} = \text{diag}\left(\hat{\mathbf{c}}_1\right) + \text{diag}\left(\hat{\mathbf{c}}_2\right), \tag{32}$$

where $\hat{\mathbf{c}} = \left[\hat{\mathbf{c}}_1^T, \hat{\mathbf{c}}_2^T\right]^T \in \mathbb{R}^{2n\times 1}$ are the (ordered) sample canonical correlations between the complex vectors $\mathbf{a} = [\mathbf{a}_1^T, \mathbf{a}_2^T]^T$ and $\mathbf{a}^*$, i.e., $\mathbf{c}$ contains the sample circularity coefficients of the complex random vector $\mathbf{a}$ [34], [38], [42]. Thus, the maximal invariant is given by the diagonal matrices $\hat{\mathbf{\Sigma}}_{\eta'}, \hat{\mathbf{\Sigma}}_{\eta''}$, or by the sample circularity coefficients $\hat{\mathbf{c}}$. Moreover, since there exists a one-to-one correspondence between $\hat{\mathbf{c}}$ and the eigenvalues of the sample $\mathbb{C}^\eta$-coherence matrix $\hat{\mathbf{\Phi}}_{\mathbb{C}^\eta}$ (which are the same as those of $\hat{\mathbf{\Phi}}_{\tilde{\mathbf{a}}}$), we can use the eigenvalues of $\hat{\mathbf{\Phi}}_{\mathbb{C}^\eta}$ (or $\hat{\mathbf{\Phi}}_{\tilde{\mathbf{a}}}$) as a maximal invariant.

3) Maximal invariant for the $\mathbb{Q}$-Properness test: This case is much more involved than the previous ones. Following the previous lines, we can see that the sufficient statistic $\hat{\mathbf{R}}_{\bar{\mathbf{x}},\bar{\mathbf{x}}}$ can be decomposed as

$$\begin{bmatrix} \widetilde{\mathbf{F}} & \mathbf{0}_{2n\times 2n} \\ \mathbf{0}_{2n\times 2n} & \widetilde{\mathbf{F}}^{(\eta')} \end{bmatrix}^{-H} \begin{bmatrix} \mathbf{I}_{2n} & \widetilde{\mathbf{\Sigma}} \\ \widetilde{\mathbf{\Sigma}} & \mathbf{I}_{2n} \end{bmatrix} \begin{bmatrix} \widetilde{\mathbf{F}} & \mathbf{0}_{2n\times 2n} \\ \mathbf{0}_{2n\times 2n} & \widetilde{\mathbf{F}}^{(\eta')} \end{bmatrix}^{-1}, \tag{33}$$

with

$$\widetilde{\mathbf{F}} = \begin{bmatrix} \mathbf{F}_1 & \mathbf{F}_\eta^{(\eta)} \\ \mathbf{F}_\eta & \mathbf{F}_1^{(\eta)} \end{bmatrix}, \qquad \widetilde{\mathbf{\Sigma}} = \begin{bmatrix} \hat{\mathbf{\Sigma}}_{\eta'} & \hat{\mathbf{\Sigma}}_{\eta''} \\ \hat{\mathbf{\Sigma}}_{\eta''} & \hat{\mathbf{\Sigma}}_{\eta'} \end{bmatrix}. \tag{34}$$

Thus, introducing the transformation $\mathbf{y}[t] = \mathbf{F}_1^H \mathbf{x}[t]$ and defining $\mathbf{G} = \mathbf{F}_1^{-1}\mathbf{F}_\eta$, we can see that a maximal invariant (under invertible *conventional* linear transformations) for the $\mathbb{Q}$-properness test is given by

$$\left\{\hat{\mathbf{\Sigma}}_{\eta'}, \hat{\mathbf{\Sigma}}_{\eta''}, \mathbf{G}\right\}, \tag{35}$$

[4]In particular, the matrix $\mathbf{F}_1$ is given by $\mathbf{F}_1 = \mathbf{R}_{\mathbf{x},\mathbf{x}}^{-1/2}\mathbf{U}$, where $\mathbf{U}$ is the unitary matrix in the factorization of $\mathbf{R}_{\mathbf{x},\mathbf{x}}^{-1/2}\mathbf{R}_{\mathbf{x},\mathbf{x}^{(\eta)}}\mathbf{R}_{\mathbf{x},\mathbf{x}}^{(\eta)\,-1/2}$ provided by Lemma 4.

TABLE II
LMPIT STATISTICS

Test	Invariances	Sample Size	LMPIT statistic
$\mathcal{H}_{\mathbb{Q}}$ vs. $\mathcal{H}_{\mathcal{I}}$	$\mathbf{F}_1^H \mathbf{x}^{(a)}$	$T \geq n$	$\left\Vert \hat{\mathbf{\Phi}}_{\mathbb{Q}} \right\Vert^2$
$\mathcal{H}_{\mathbb{C}^\eta}$ vs. $\mathcal{H}_{\mathcal{I}}$	$\mathbf{F}_1^H \mathbf{x} + \mathbf{F}_\eta^H \mathbf{x}^{(\eta)}$	$T \geq 2n$	$\left\Vert \hat{\mathbf{\Phi}}_{\mathbb{C}^\eta} \right\Vert^2$
$\mathcal{H}_{\mathbb{Q}}$ vs. $\mathcal{H}_{\mathbb{C}^\eta}$	$\mathbf{F}_1^H \mathbf{x}$	$T \geq n$	$\left\Vert \hat{\mathbf{\Phi}}_{\mathbb{R}^\eta} \right\Vert^2$

i.e., $2n$ (ordered) sample canonical correlations and a quaternion matrix $\mathbf{G} \in \mathbb{H}^{n\times n}$, which is unambiguously specified up to individual products of its rows by unit quaternions in the plane $\{1, \eta\}$.

Obviously, the above maximal invariant does not have the nice form of those derived in the previous cases. In particular, there is not a one-to-one correspondence between the maximal invariant and the eigenvalues of the sample $\mathbb{Q}$-coherence matrix $\hat{\mathbf{\Phi}}_{\mathbb{Q}}$. Moreover, although the consideration of the invariance under quaternion rotations $\mathbf{y} = \mathbf{x}^{(a)}$ could introduce a slight reduction in the degrees of freedom of the quaternion matrix $\mathbf{G}$, it is not enough for providing such an elegant maximal invariant.

B. Locally Most Powerful Invariant Test (LMPITs)

Interestingly, although the derivation of the maximal invariant in the case of the $\mathbb{Q}$-properness test seems to be very complicated, the three LMPITs can be directly obtained with the help of the Wijsman's theorem [50], [51], [54], [63], [64], whose details are provided in the next section. Here, we present the three LMPITs and analyze some of their properties and connections with related problems.

The LMPITs, which are summarized in Table II, reject the null (proper) hypothesis for high values of the test statistics. As can be seen, all the test statistics are functions of the eigenvalues of the associated sample coherence matrices, which was obvious in the cases of testing $\mathcal{H}_{\mathbb{C}^\eta}$ versus $\mathcal{H}_{\mathcal{I}}$, and $\mathcal{H}_{\mathbb{Q}}$ versus $\mathcal{H}_{\mathbb{C}^\eta}$, but it was not clear (although intuitively appealing) for the $\mathbb{Q}$-properness test. Moreover, it can be seen that the LMPITs require a lower number of vector samples than their GLRT counterparts, and it is also easy to prove that the LMPIT statistics never exceed $16n^2$.

As previously noted, the GLRTs for the $\mathbb{C}^\eta$-properness of $\mathbf{x}$ and the complex properness of $\mathbf{a} = [\mathbf{a}_1^T, \mathbf{a}_2^T]^T$ coincide [40], [42], [43], and the same happens with the LMPITs. In particular, the LMPIT statistic for the complex properness of $\mathbf{a}$ can be written as $\left\Vert \hat{\mathbf{\Phi}}_{\tilde{\mathbf{a}}} \right\Vert^2$ [40], [43], and due to eq. (27) and the unitarity of $\mathbf{L}$, it is clear that $\left\Vert \hat{\mathbf{\Phi}}_{\mathbb{C}^\eta} \right\Vert^2 = \left\Vert \hat{\mathbf{\Phi}}_{\tilde{\mathbf{a}}} \right\Vert^2$. Thus, as suggested by Lemma 5, the $\mathbb{C}^\eta$-properness test reduces to the problem of testing for the joint properness of the complex vectors $\mathbf{a}_1$, $\mathbf{a}_2$ in the Cayley-Dickson representation $\mathbf{x} = \mathbf{a}_1 + \eta''\mathbf{a}_2$.

The particular case of scalar quaternions $x \in \mathbb{H}$ also provides some interesting insights.[5] Specifically, in the scalar

[5]The careful reader will also note that in the scalar case the LMPITs require $T > 1$.

case we have $\hat{\mathbf{R}}_{\bar{\mathbf{x}},\bar{\mathbf{x}}} = 4\mathbf{T}_1\hat{\mathbf{R}}_{\mathbf{r},\mathbf{r}}\mathbf{T}_1^H$ and $\hat{\mathbf{D}}_{\mathbb{Q}} = \text{Tr}(\hat{\mathbf{R}}_{\mathbf{r},\mathbf{r}})\mathbf{I}_4$, and therefore the $\mathbb{Q}$-properness GLRT and LMPIT statistics can be rewritten as

$$\hat{\mathcal{P}}_{\mathbb{Q}} = -2\ln\frac{|\hat{\mathbf{R}}_{\mathbf{r},\mathbf{r}}|^{1/4}}{\text{Tr}(\hat{\mathbf{R}}_{\mathbf{r},\mathbf{r}})/4}, \quad (36)$$

$$\left\|\hat{\mathbf{\Phi}}_{\mathbb{Q}}\right\|^2 = 16\frac{\left\|\hat{\mathbf{R}}_{\mathbf{r},\mathbf{r}}\right\|^2}{\text{Tr}^2(\hat{\mathbf{R}}_{\mathbf{r},\mathbf{r}})}, \quad (37)$$

which coincide with the GLRT [56] and LMPIT [57] statistics for testing the sphericity of the real vector $\mathbf{r} = [r_1, r_\eta, r_{\eta'}, r_{\eta''}]^T \in \mathbb{R}^{4\times 1}$. Here, we must note an important difference with the scalar complex case [40], [42], [43], which consists in the fact that in the quaternion scalar case, the $\mathbb{Q}$-properness (and $\mathbb{C}^\eta$-properness) GLRT and LMPIT do not coincide. Finally, in the case of testing for the $\mathbb{Q}$-properness of the $\mathbb{C}^\eta$-proper quaternion $x \in \mathbb{H}$ ($\mathcal{H}_{\mathbb{Q}}$ versus $\mathcal{H}_{\mathbb{C}^\eta}$), the GLRT and LMPIT coincide, and they amount to testing for the sphericity of the proper complex vector $\mathbf{a} = [a_1, a_2]^T$.

Regarding the distribution of the LMPIT statistics, we must note that, even in the complex [43] or the quaternion scalar case [58], this is a very difficult problem beyond the scope of this paper. However, analogously to the GLRT case [19], [47], the invariance principle allows us to obtain the distributions of the tests statistics under the null (proper) hypothesis by means of simulations. As we will see in Section VI, the distributions can be obtained from Gaussian data with $\mathbf{R}_{\bar{\mathbf{x}},\bar{\mathbf{x}}} = \mathbf{I}_{4n}$, and we only need to tabulate the results for different values of n and T. On the other hand, if we focus on the null (proper) hypothesis and asymptotically large sample sizes ($T \to \infty$), we will have sample coherence matrices close to the identity, which allows us to relate the GLRT and LMPIT statistics as

$$\hat{\mathcal{P}}_{\mathbb{Q}} \overset{\mathcal{H}_{\mathbb{Q}}}{\simeq} \frac{1}{4}\left\|\hat{\mathbf{\Phi}}_{\mathbb{Q}}\right\|^2 - n, \quad (38)$$

$$\hat{\mathcal{P}}_{\mathbb{C}^\eta} \overset{\mathcal{H}_{\mathbb{C}^\eta}}{\simeq} \frac{1}{4}\left\|\hat{\mathbf{\Phi}}_{\mathbb{C}^\eta}\right\|^2 - n, \quad (39)$$

$$\hat{\mathcal{P}}_{\mathbb{R}^\eta} \overset{\mathcal{H}_{\mathbb{Q}}}{\simeq} \frac{1}{4}\left\|\hat{\mathbf{\Phi}}_{\mathbb{R}^\eta}\right\|^2 - n, \quad (40)$$

where the notation $\simeq^{\mathcal{H}}$ means approximated under the hypothesis $\mathcal{H}$ and $T \to \infty$. Thus, applying the Wilks' theorem to the GLRT statistics [19], [47], [65], we can obtain the approximated distributions

$$\frac{T}{2}\left(\left\|\hat{\mathbf{\Phi}}_{\mathbb{Q}}\right\|^2 - 4n\right) \overset{\mathcal{H}_{\mathbb{Q}}}{\sim} \chi^2_{d_{\mathbb{Q}}}, \quad (41)$$

$$\frac{T}{2}\left(\left\|\hat{\mathbf{\Phi}}_{\mathbb{C}^\eta}\right\|^2 - 4n\right) \overset{\mathcal{H}_{\mathbb{C}^\eta}}{\sim} \chi^2_{d_{\mathbb{C}^\eta}}, \quad (42)$$

$$\frac{T}{2}\left(\left\|\hat{\mathbf{\Phi}}_{\mathbb{R}^\eta}\right\|^2 - 4n\right) \overset{\mathcal{H}_{\mathbb{Q}}}{\sim} \chi^2_{d_{\mathbb{R}^\eta}}, \quad (43)$$

where $\sim^{\mathcal{H}}$ means approximated distribution under the hypothesis $\mathcal{H}$ and $T \to \infty$, χ^2_d is a central chi-square distribution with d degrees of freedom and [19], [47]

$$d_{\mathbb{R}^\eta} = \frac{1}{2}d_{\mathbb{C}^\eta} = \frac{1}{3}d_{\mathbb{Q}} = n(2n+1). \quad (44)$$

Interestingly, due to the finite support $[0, 16n^2]$ of the LMPIT statistics, the Wilks' approximation (or any other approximation with infinite support $[0, \infty]$) is *conservative*, which most statisticians consider preferable to the alternative, described as *liberal* tests. In other words, for sufficiently low nominal levels of the false alarm probability P_f, which is defined as the probability of rejecting the null (proper) hypothesis when it is true, the actual false alarm probability of the LMPITs based on the Wilks' approximation is lower than its nominal level.

Finally, we must note that the LMPIT statistics do not satisfy a relationship similar to that in eq. (25). However, from eqs. (38)-(40), we can write

$$\left\|\hat{\mathbf{\Phi}}_{\mathbb{Q}}\right\|^2 \overset{\mathcal{H}_{\mathbb{Q}}}{\simeq} \left\|\hat{\mathbf{\Phi}}_{\mathbb{C}^\eta}\right\|^2 + \left\|\hat{\mathbf{\Phi}}_{\mathbb{R}^\eta}\right\|^2 - 4n. \quad (45)$$

Apart from the derivation of the LMPITs, it is important to consider the existence of uniformly most powerful invariant tests (UMPITs). From the derivation in the next section, it is easy to conclude that the only binary hypothesis testing problem for which a UMPIT exists is the problem of testing whether the $\mathbb{C}^\eta$-proper scalar quaternion $x \in \mathbb{H}$ is also $\mathbb{Q}$-proper. As previously pointed out, in this case the GLRT and LMPIT coincide, and they are also equivalent to the UMPIT, whose statistic is given by the absolute value of the sample correlation coefficient (or canonical correlation [24], [62]) between x and $x^{(\eta)}$. This is not a surprising result because we already knew that, for complex vectors, the UMPIT only exists in the scalar case [40], [43]. Therefore, since the $\mathbb{C}^\eta$-properness test is equivalent to the problem of testing for the complex properness of $\mathbf{a} = [\mathbf{a}_1^T, \mathbf{a}_2^T]^T$, it is clear that there does not exist a $\mathbb{C}^\eta$-properness UMPIT. On the other hand, the $\mathbb{Q}$-properness test can be seen as a problem more complicated than the $\mathbb{C}^\eta$-properness test, and therefore we should not expect the existence of a $\mathbb{Q}$-properness UMPIT, which is corroborated by the results in the next section.

V. DERIVATION OF THE LMPITS

As previously pointed out, the key ingredient for the derivation of the LMPITs is provided by the Wijsman's theorem [50], [51], [54], [63], [64], which allows us to obtain the ratio between the densities of the maximal invariants, not only without the knowledge of the densities, but also without an explicit expression for the maximal invariants. The key idea of the Wijsman's theorem consists in integrating over the group describing the problem invariance. In particular, if we consider a binary hypothesis testing problem with observations[6] $\mathbf{x} \in \mathcal{S}$, and invariant under the group $\mathcal{G}$ of linear transformations $\mathbf{y} = \mathbf{G}\mathbf{x}$ ($\mathbf{G} \in \mathcal{G}$), the Wijsman's theorem states that the ratio $R(\hat{\mathbf{m}})$ between the densities of the maximal invariant $\hat{\mathbf{m}}$ is given by

$$R(\hat{\mathbf{m}}) = \frac{p(\hat{\mathbf{m}}; \mathcal{H}_1)}{p(\hat{\mathbf{m}}; \mathcal{H}_0)} = \frac{\int_{\mathcal{G}} p(\mathbf{G}\mathbf{x}; \mathcal{H}_1)\,|\mathbf{G}|\,d\mathbf{G}}{\int_{\mathcal{G}} p(\mathbf{G}\mathbf{x}; \mathcal{H}_0)\,|\mathbf{G}|\,d\mathbf{G}}, \quad (46)$$

where $p(\cdot; \mathcal{H}_0)$ and $p(\cdot; \mathcal{H}_1)$ denote the densities of $\hat{\mathbf{m}}$ or $\mathbf{x}$ under the null and alternative hypotheses, $\mathbf{G}$ is the Jacobian of the transformation $\mathbf{y} = \mathbf{G}\mathbf{x}$, and $d\mathbf{G}$ is an invariant group

[6]Here, the vector $\mathbf{x}$ denotes the observations (or a sufficient statistic) of a general testing problem, and should not be interpreted as a quaternion random vector.

measure, which in this paper can be considered as the usual Lebesgue measure.

The idea of integrating over the group describing the problem invariances was first introduced by Stein [55], and the conditions for the validity of the Wijsman's theorem have been studied by several authors [41], [50], [52], [53], [63], [66], [67]. For our purposes, it is sufficient to consider the simplest conditions [63], which state that $\mathcal{G}$ is a Lie group, and $\mathcal{S}$ is a linear Cartan $\mathcal{G}$-space, i.e., it is a nonempty open subset of the Euclidean space such that, for every $\mathbf{x} \in \mathcal{S}$, there exists a neighborhood $\mathcal{V}$ for which the closure of $\{\mathbf{G} \in \mathcal{G} : \mathbf{G}\mathcal{V} \cap \mathcal{V} \neq \emptyset\}$ is compact.

The derivation of the three LMPITs follows the lines in [40] for the complex case (see also [41]), and it is divided in three main parts. Firstly, the density ratio is obtained by direct application of the Wijsman's theorem; secondly, the density ratio is simplified for the case of very close null (proper) and alternative hypotheses; finally, some straightforward algebra allows us to solve the integrals and obtain the LMPIT statistics.

For the sake of space, here we focus on the most complicated case, which is that of the $\mathbb{Q}$-properness LMPIT. The other two LMPITs are briefly commented in Subsection V-D, but its complete derivation is left as an exercise for the interested readers.

A. First Step: Ratio of Maximal Invariant Densities

As we have seen, the $\mathbb{Q}$-properness test is invariant under the group $\mathcal{G}_{\mathbb{Q}}$ of conventional linear transformations and quaternion rotations $\mathbf{y} = \mathbf{F}_1^H \mathbf{x}^{(a)}$. However, in order to simplify the derivation of the LMPIT, we can avoid redundancies focusing on right-Clifford translations, i.e., right products by a unit quaternion a ($|a| = 1$), and conventional linear transformations

$$\mathbf{y} = \mathbf{F}_1^H \mathbf{x} a. \tag{47}$$

Thus, taking into account the isomorphism between $\mathbb{H}^{n\times 1}$ and $\mathbb{R}^{4n\times 1}$, it is easy to prove that $\mathcal{G}_{\mathbb{Q}}$ is a Lie group and, for $T \geq n$ quaternion vector observations $\mathbf{x}[t]$ ($t = 0, \ldots, T-1$), the observation space $\mathcal{S} = \mathbb{H}^{n\times T}$ is a linear Cartan $\mathcal{G}_{\mathbb{Q}}$-space. Therefore, the direct application of the Wijsman's theorem allows us to write

$$\begin{aligned} R_{\mathbb{Q}} &= R(\hat{\boldsymbol{\Phi}}_{\mathbb{Q}}) = R(\hat{\mathbf{m}}_{\mathbb{Q}}) = \frac{p(\hat{\mathbf{m}}_{\mathbb{Q}}; \mathcal{H}_{\mathcal{I}})}{p(\hat{\mathbf{m}}_{\mathbb{Q}}; \mathcal{H}_{\mathbb{Q}})} \\ &= \frac{\int_{\mathbf{F}_1} \int_{|a|=1} \left(\prod_{t=0}^{T-1} p_{\mathcal{H}_{\mathcal{I}}}(\mathbf{F}_1^H \mathbf{x}[t] a) \right) \left| \mathbf{F}_1^H \mathbf{F}_1 \right|^{2T} d\mathbf{F}_1 da}{\int_{\mathbf{F}_1} \int_{|a|=1} \left(\prod_{t=0}^{T-1} p_{\mathcal{H}_{\mathbb{Q}}}(\mathbf{F}_1^H \mathbf{x}[t] a) \right) \left| \mathbf{F}_1^H \mathbf{F}_1 \right|^{2T} d\mathbf{F}_1 da}, \end{aligned} \tag{48}$$

where we have used the fact that the Jacobian of the right-Clifford translation is one, and it has been explicitly stated that the density ratio of the maximal invariant $\hat{\mathbf{m}}_{\mathbb{Q}}$ can be written as a function of the sample $\mathbb{Q}$-coherence matrix $\hat{\boldsymbol{\Phi}}_{\mathbb{Q}}$.

Denote now the augmented covariance matrix under the two hypotheses as $\mathbf{R}_{\bar{\mathbf{x}},\bar{\mathbf{x}}}(\mathcal{H}_{\mathbb{Q}})$ and $\mathbf{R}_{\bar{\mathbf{x}},\bar{\mathbf{x}}}(\mathcal{H}_{\mathcal{I}})$. Then, thanks to the invariance of the testing problem under invertible quaternion linear transformations, we can assume without loss of generality that $\mathbf{R}_{\bar{\mathbf{x}},\bar{\mathbf{x}}}^{-1}(\mathcal{H}_{\mathcal{I}}) = \mathbf{R}_{\bar{\mathbf{z}},\bar{\mathbf{z}}}$, with

$$\mathbf{R}_{\bar{\mathbf{z}},\bar{\mathbf{z}}} = \begin{bmatrix} \mathbf{I}_n & \mathbf{R}_{\mathbf{z},\mathbf{z}^{(\eta)}} & \mathbf{R}_{\mathbf{z},\mathbf{z}^{(\eta')}} & \mathbf{R}_{\mathbf{z},\mathbf{z}^{(\eta'')}} \\ \mathbf{R}_{\mathbf{z},\mathbf{z}^{(\eta)}}^{(\eta)} & \mathbf{I}_n & \mathbf{R}_{\mathbf{z},\mathbf{z}^{(\eta'')}}^{(\eta)} & \mathbf{R}_{\mathbf{z},\mathbf{z}^{(\eta')}}^{(\eta)} \\ \mathbf{R}_{\mathbf{z},\mathbf{z}^{(\eta')}}^{(\eta')} & \mathbf{R}_{\mathbf{z},\mathbf{z}^{(\eta'')}}^{(\eta')} & \mathbf{I}_n & \mathbf{R}_{\mathbf{z},\mathbf{z}^{(\eta)}}^{(\eta')} \\ \mathbf{R}_{\mathbf{z},\mathbf{z}^{(\eta'')}}^{(\eta'')} & \mathbf{R}_{\mathbf{z},\mathbf{z}^{(\eta')}}^{(\eta'')} & \mathbf{R}_{\mathbf{z},\mathbf{z}^{(\eta)}}^{(\eta'')} & \mathbf{I}_n \end{bmatrix}, \tag{49}$$

where the augmented quaternion vector $\bar{\mathbf{z}}$ is defined, under the hypothesis $\mathcal{H}_{\mathcal{I}}$, as $\bar{\mathbf{z}} = \mathbf{R}_{\bar{\mathbf{x}},\bar{\mathbf{x}}}^{-1}(\mathcal{H}_{\mathcal{I}})\bar{\mathbf{x}}$. Thus, using the expression in (18) for the quaternion Gaussian distribution we get[7]

$$R_{\mathbb{Q}} \propto \frac{\int_{\overline{\mathbf{F}}} \int_{|a|=1} \left| \overline{\mathbf{F}} \right|^T e^{-\frac{T}{2}\Re\left[\mathrm{Tr}\left(\mathbf{R}_{\bar{\mathbf{x}},\bar{\mathbf{x}}}^{-1}(\mathcal{H}_{\mathcal{I}})\overline{\mathbf{F}}^H \hat{\mathbf{R}}_{\bar{\mathbf{y}},\bar{\mathbf{y}}}\overline{\mathbf{F}}\right)\right]} d\overline{\mathbf{F}} da}{\int_{\overline{\mathbf{F}}} \int_{|a|=1} \left| \overline{\mathbf{F}} \right|^T e^{-\frac{T}{2}\Re\left[\mathrm{Tr}\left(\mathbf{R}_{\bar{\mathbf{x}},\bar{\mathbf{x}}}^{-1}(\mathcal{H}_{\mathbb{Q}})\overline{\mathbf{F}}^H \hat{\mathbf{R}}_{\bar{\mathbf{y}},\bar{\mathbf{y}}}\overline{\mathbf{F}}\right)\right]} d\overline{\mathbf{F}} da}, \tag{50}$$

where $\overline{\mathbf{F}}$ is the (block-diagonal) full-widely linear operator associated to the conventional linear transformation, $\propto$ means equality up to constant terms (0 in this case) and a scaling factor ($\left|\mathbf{R}_{\bar{\mathbf{x}},\bar{\mathbf{x}}}^{-1}(\mathcal{H}_{\mathcal{I}})\mathbf{R}_{\bar{\mathbf{x}},\bar{\mathbf{x}}}(\mathcal{H}_{\mathbb{Q}})\right|^{\frac{T}{2}}$) that does not depend on the observations, and $\hat{\mathbf{R}}_{\bar{\mathbf{y}},\bar{\mathbf{y}}}$ denotes the augmented sample covariance matrix of the vector $\mathbf{y} = \mathbf{x}a$. Moreover, taking into account the invariance of $R_{\mathbb{Q}}$ under invertible linear transformations, we can always introduce a transformation such that the sample covariance matrix of $\mathbf{x}$ is $\hat{\mathbf{R}}_{\mathbf{x},\mathbf{x}} = \mathbf{I}_n$, which also implies $\hat{\mathbf{R}}_{\mathbf{y},\mathbf{y}} = \mathbf{I}_n$, and allows us to replace $\hat{\mathbf{R}}_{\bar{\mathbf{x}},\bar{\mathbf{x}}}$ by $\hat{\boldsymbol{\Phi}}_{\mathbb{Q}}$ without loss of generality. Thus, we have

$$R_{\mathbb{Q}} \propto \frac{\int_{\overline{\mathbf{F}}} \int_{|a|=1} \left| \overline{\mathbf{F}} \right|^T e^{-\frac{T}{2}\Re\left[\mathrm{Tr}\left(\mathbf{R}_{\bar{\mathbf{x}},\bar{\mathbf{x}}}^{-1}(\mathcal{H}_{\mathcal{I}})\overline{\mathbf{F}}^H \hat{\mathbf{R}}_{\bar{\mathbf{y}},\bar{\mathbf{y}}}\overline{\mathbf{F}}\right)\right]} d\overline{\mathbf{F}} da}{\int_{\overline{\mathbf{F}}} \int_{|a|=1} \left| \overline{\mathbf{F}} \right|^T e^{-\frac{T}{2}\Re\left[\mathrm{Tr}\left(\mathbf{R}_{\bar{\mathbf{x}},\bar{\mathbf{x}}}^{-1}(\mathcal{H}_{\mathbb{Q}})\overline{\mathbf{F}}^H \overline{\mathbf{F}}\right)\right]} d\overline{\mathbf{F}} da}, \tag{51}$$

and since the denominator does not depend on the observations we can write

$$R_{\mathbb{Q}} \propto \int_{\overline{\mathbf{F}}} \int_{|a|=1} \left| \overline{\mathbf{F}} \right|^T e^{-\frac{T}{2}\Re\left[\mathrm{Tr}\left(\mathbf{R}_{\bar{\mathbf{x}},\bar{\mathbf{x}}}^{-1}(\mathcal{H}_{\mathcal{I}})\overline{\mathbf{F}}^H \hat{\mathbf{R}}_{\bar{\mathbf{y}},\bar{\mathbf{y}}}\overline{\mathbf{F}}\right)\right]} d\overline{\mathbf{F}} da, \tag{52}$$

or equivalently

$$R_{\mathbb{Q}} \propto \int_{\mathbf{F}_1} \int_{|a|=1} \left| \mathbf{F}_1^H \mathbf{F}_1 \right|^{2T} e^{-2T\mathrm{Tr}\left(\mathbf{F}_1^H \mathbf{F}_1\right)} e^{-2T\hat{\theta}_a} d\mathbf{F}_1 da, \tag{53}$$

where $\hat{\theta}_a = \hat{\theta}_{\eta,a} + \hat{\theta}_{\eta',a} + \hat{\theta}_{\eta'',a}$, and

$$\hat{\theta}_{\nu,a} = \Re\left[\mathrm{Tr}\left(\mathbf{R}_{\mathbf{z},\mathbf{z}^{(\nu)}}^{(\nu)} \mathbf{F}_1^H \hat{\mathbf{R}}_{\mathbf{y},\mathbf{y}^{(\nu)}} \mathbf{F}_1^{(\nu)}\right)\right], \tag{54}$$

for all pure unit quaternions ν.

B. Second Step: Approximation for $\mathbf{R}_{\bar{\mathbf{z}},\bar{\mathbf{z}}} \approx \mathbf{I}_{4n}$

The ratio of maximal invariant densities in (53) provides the test statistic for the most powerful invariant test (MPI). However, the density ratio depends on the unknown parameters in $\mathbf{R}_{\bar{\mathbf{z}},\bar{\mathbf{z}}}$ through the coefficient $\hat{\theta}_a$, which precludes the obtention

[7]Note that the real operator in the exponents are due to the non-commutativity of the quaternion product. Alternatively, we could write $\Re\left[\mathrm{Tr}\left(\mathbf{R}_{\bar{\mathbf{x}},\bar{\mathbf{x}}}^{-1}\overline{\mathbf{F}}^H \hat{\mathbf{R}}_{\bar{\mathbf{y}},\bar{\mathbf{y}}}\overline{\mathbf{F}}\right)\right] = \mathrm{Tr}\left(\mathbf{R}_{\bar{\mathbf{x}},\bar{\mathbf{x}}}^{-1/2}\overline{\mathbf{F}}^H \hat{\mathbf{R}}_{\bar{\mathbf{y}},\bar{\mathbf{y}}}\overline{\mathbf{F}}\mathbf{R}_{\bar{\mathbf{x}},\bar{\mathbf{x}}}^{-1/2}\right)$.

TABLE III
EFFECT OF RIGHT-CLIFFORD TRANSLATIONS ON THE COEFFICIENTS $\hat{\theta}_{\nu,a}$

Right-Clifford Translations	Coefficients		
$\mathbf{x}[t]$	$\hat{\theta}_{\eta,a}$	$\hat{\theta}_{\eta',a}$	$\hat{\theta}_{\eta'',a}$
$\mathbf{x}[t]\eta$	$\hat{\theta}_{\eta,a}$	$-\hat{\theta}_{\eta',a}$	$-\hat{\theta}_{\eta'',a}$
$\mathbf{x}[t]\eta'$	$-\hat{\theta}_{\eta,a}$	$\hat{\theta}_{\eta',a}$	$-\hat{\theta}_{\eta'',a}$
$\mathbf{x}[t]\eta''$	$-\hat{\theta}_{\eta,a}$	$-\hat{\theta}_{\eta',a}$	$\hat{\theta}_{\eta'',a}$

of a uniformly most powerful invariant test (UMPIT). Here, we focus on the challenging case in which the null ($\mathcal{H}_{\mathbb{Q}}$) and alternative ($\mathcal{H}_{\mathcal{I}}$) hypotheses are very close. That is, we assume $\mathbf{R}_{\bar{\mathbf{z}},\bar{\mathbf{z}}} \approx \mathbf{I}_{4n}$ and apply a second-order Taylor's series approximation to the second exponential in (53)

$$e^{-2T\hat{\theta}_a} \approx 1 - 2T\hat{\theta}_a + 2T^2\hat{\theta}_a^2, \tag{55}$$

which yields

$$R_{\mathbb{Q}} \propto \int_{\mathbf{F}_1}\int_{|a|=1} \left|\mathbf{F}_1^H\mathbf{F}_1\right|^{2T} e^{-2T\mathrm{Tr}\left(\mathbf{F}_1^H\mathbf{F}_1\right)} \left(T\hat{\theta}_a^2 - \hat{\theta}_a\right) d\mathbf{F}_1 da. \tag{56}$$

Now, taking into account $\mathbf{y} = \mathbf{x}a$, and using the result in Lemma 2, it is easy to prove that $\hat{\mathbf{R}}_{\mathbf{y},\mathbf{y}^{(\nu)}} = \hat{\mathbf{R}}_{\mathbf{x},\mathbf{x}^{(\nu^{(a)})}}\nu^{(a)}\nu^*$ and, as a consequence

$$\hat{\theta}_{\nu,a} = \Re\left[\mathrm{Tr}\left(\mathbf{R}_{\mathbf{z},\mathbf{z}^{(\nu)}}\nu^*\mathbf{F}_1^H\hat{\mathbf{R}}_{\mathbf{x},\mathbf{x}^{(\nu^{(a)})}}\nu^{(a)}\mathbf{F}_1\right)\right]. \tag{57}$$

This allows us to obtain the results in Table III, which summarizes the effects on the coefficients $\hat{\theta}_{\nu,a}$, of some right-Clifford translations $\mathbf{x}[t]\nu$ of the observations. Furthermore, thanks to the invariance under right-Clifford translations, the vector observations $\mathbf{x}[t]$ can be replaced by $\mathbf{x}[t]\nu$ without changing the density ratio $R_{\mathbb{Q}}$. Thus, averaging the expressions in (56) for the right-Clifford translations in Table III, we can get rid of the linear and cross-product terms and obtain

$$R_{\mathbb{Q}} \propto \mu_\eta + \mu_{\eta'} + \mu_{\eta''}, \tag{58}$$

where

$$\mu_\nu = \int_{\mathbf{F}_1}\int_{|a|=1} \hat{\theta}_{\nu,a}^2 \left|\mathbf{F}_1^H\mathbf{F}_1\right|^{2T} e^{-2T\mathrm{Tr}\left(\mathbf{F}_1^H\mathbf{F}_1\right)} d\mathbf{F}_1 da. \tag{59}$$

C. Third Step: Integrals for the Coefficients μ_ν

At this point, the problem reduces to solve the integrals in (59). In order to do that, let us start by focusing on the coefficients $\hat{\theta}_{\nu,a}$, and using the unitary factorization (see Lemma 4) of the matrices $\mathbf{R}_{\mathbf{z},\mathbf{z}^{(\nu)}}$ and $\hat{\mathbf{R}}_{\mathbf{x},\mathbf{x}^{(\nu^{(a)})}}$ to write

$$\mathbf{R}_{\mathbf{z},\mathbf{z}^{(\nu)}}\nu^* = \mathbf{Q}\boldsymbol{\Lambda}\nu\mathbf{Q}^H \qquad \hat{\mathbf{R}}_{\mathbf{x},\mathbf{x}^{(\nu^{(a)})}}\nu^{(a)} = \hat{\mathbf{Q}}\hat{\boldsymbol{\Lambda}}\nu\hat{\mathbf{Q}}^H, \tag{60}$$

where $\mathbf{Q}, \hat{\mathbf{Q}} \in \mathbb{H}^{n\times n}$ are unitary matrices, and $\boldsymbol{\Lambda}, \hat{\boldsymbol{\Lambda}} \in \mathbb{R}^{n\times n}$ are diagonal matrices with the singular values $\lambda_k, \hat{\lambda}_k \in \mathbb{R}$ in their diagonal entries.

Due to the unitarity of $\mathbf{Q}, \hat{\mathbf{Q}}$, these matrices can be absorbed in $\mathbf{F}_1$ without changing the integral in (59), which yields

$$\hat{\theta}_{\nu,a} = \sum_{k=1}^{n}\sum_{l=1}^{n} \lambda_k\hat{\lambda}_l \Re\left(\nu f_{k,l}^*\nu f_{k,l}\right), \tag{61}$$

where $f_{k,l}$ is the entry in the k-th row and l-th column of $\mathbf{F}_1$. Thus, (59) can be rewritten as

$$\begin{aligned}\mu_\nu &= \varepsilon \sum_{k=1}^{n}\lambda_k^2 \int_{|a|=1}\sum_{k=1}^{n}\hat{\lambda}_k^2 da \\ &= \varepsilon \left\|\mathbf{R}_{\mathbf{z},\mathbf{z}^{(\nu)}}\right\|^2 \int_{|a|=1}\left\|\hat{\mathbf{R}}_{\mathbf{x},\mathbf{x}^{(\nu^{(a)})}}\right\|^2 da,\end{aligned} \tag{62}$$

where

$$\varepsilon = \int_{\mathbf{F}_1} \left|\mathbf{F}_1^H\mathbf{F}_1\right|^{2T} e^{-2T\mathrm{Tr}\left(\mathbf{F}_1^H\mathbf{F}_1\right)} \left[\Re\left(\nu f_{k,l}^*\nu^{(a)}f_{k,l}\right)\right]^2 d\mathbf{F}_1. \tag{63}$$

Furthermore, a simple change of the integration variable allows us to conclude that the integral in (62) does not depend on ν, i.e.,

$$\beta = \beta(\nu) = \int_{|a|=1}\left\|\hat{\mathbf{R}}_{\mathbf{x},\mathbf{x}^{(\nu^{(a)})}}\right\|^2 da, \tag{64}$$

which yields $\beta = \left(\beta(\eta)+\beta(\eta')+\beta(\eta'')\right)/3$, and using Lemmas 2 and 3 we can write $\beta \propto \left\|\hat{\mathbf{R}}_{\bar{\mathbf{x}},\bar{\mathbf{x}}}\right\|^2$. Finally, taking into account that we have introduced a linear transformation such that $\hat{\mathbf{R}}_{\mathbf{x},\mathbf{x}} = \mathbf{I}_n$, and combining eqs. (58), (59) and (62), we obtain the $\mathbb{Q}$-properness LMPIT statistic, i.e., the ratio of maximal invariant densities for close hypotheses $\mathcal{H}_{\mathbb{Q}}$ and $\mathcal{H}_{\mathcal{I}}$ is

$$R_{\mathbb{Q}} \propto \left\|\hat{\boldsymbol{\Phi}}_{\mathbb{Q}}\right\|^2. \tag{65}$$

D. Derivation of the Remaining LMPITs

As previously noted, the derivation of the remaining LMPITs follows the lines in the previous subsections. Here, the principal details are briefly summarized.

1) Ratio of Maximal Invariant Densities: The group of transformations for the $\mathbb{C}^\eta$-properness test (respectively for the problem of testing $\mathcal{H}_{\mathbb{Q}}$ versus $\mathcal{H}_{\mathbb{C}}^\eta$) consists in invertible semi-widely linear transformations $\mathcal{G}_{\mathbb{C}^\eta}$ (resp. invertible conventional linear transformations $\mathcal{G}_{\mathbb{R}^\eta}$). For the application of the Wijsman's theorem, we need the observation space $\mathcal{S}$ to be a linear Cartan $\mathcal{G}_{\mathbb{C}^\eta}$- (resp. $\mathcal{G}_{\mathbb{R}^\eta}$-) space, which is satisfied iff $T \geq 2n$ (resp. $T \geq n$). Under these conditions, the counterparts of eqs. (53) and (54) for the $\mathbb{C}^\eta$-properness test are

$$R_{\mathbb{C}^\eta} = \frac{p(\hat{\mathbf{m}}_{\mathbb{C}^\eta};\mathcal{H}_{\mathcal{I}})}{p(\hat{\mathbf{m}}_{\mathbb{C}^\eta};\mathcal{H}_{\mathbb{C}^\eta})} \propto \int_{\widetilde{\mathbf{F}}} \left|\widetilde{\mathbf{F}}^H\widetilde{\mathbf{F}}\right|^T e^{-T\mathrm{Tr}\left(\widetilde{\mathbf{F}}^H\widetilde{\mathbf{F}}\right)} e^{-T\hat{\theta}_{\mathbb{C}^\eta}} d\widetilde{\mathbf{F}}, \tag{66}$$

$$\hat{\theta}_{\mathbb{C}^\eta} = \Re\left[\mathrm{Tr}\left(\mathbf{R}_{\tilde{\mathbf{z}},\tilde{\mathbf{z}}^{(\eta')}}^{(\eta')}\widetilde{\mathbf{F}}^H\hat{\mathbf{R}}_{\tilde{\mathbf{x}},\tilde{\mathbf{x}}^{(\eta')}}\widetilde{\mathbf{F}}^{(\eta')}\right)\right], \tag{67}$$

where the invertible semi-widely linear operator $\widetilde{\mathbf{F}}$ is defined in (34), $\tilde{\mathbf{x}} = [\mathbf{x}^T, \mathbf{x}^{(\eta)^T}]^T$ was defined in (26) as the semi-augmented quaternion vector, $\bar{\mathbf{z}} = [\tilde{\mathbf{z}}^T, \tilde{\mathbf{z}}^{(\eta')^T}]^T$, and we have introduced an invertible semi-widely linear transformation such that $\mathbf{R}_{\tilde{\mathbf{z}},\tilde{\mathbf{z}}} = \mathbf{I}_{2n}$, with $\mathbf{R}_{\bar{\mathbf{z}},\bar{\mathbf{z}}} = \mathbf{R}_{\bar{\mathbf{x}},\bar{\mathbf{x}}}^{-1}(\mathcal{H}_{\mathcal{I}})$. Analogously, for the problem of testing $\mathcal{H}_{\mathbb{Q}}$ versus $\mathcal{H}_{\mathbb{C}^\eta}$

$$\begin{aligned}R_{\mathbb{R}^\eta} &= \frac{p(\hat{\mathbf{m}}_{\mathbb{R}^\eta};\mathcal{H}_{\mathbb{C}^\eta})}{p(\hat{\mathbf{m}}_{\mathbb{R}^\eta};\mathcal{H}_{\mathbb{Q}})} \\ &\propto \int_{\mathbf{F}_1}\left|\mathbf{F}_1^H\mathbf{F}_1\right|^{2T} e^{-2T\mathrm{Tr}\left(\mathbf{F}_1^H\mathbf{F}_1\right)} e^{-2T\hat{\theta}_{\mathbb{R}^\eta}} d\mathbf{F}_1,\end{aligned} \tag{68}$$

$$\hat{\theta}_{\mathbb{R}^\eta} = \Re\left[\mathrm{Tr}\left(\mathbf{R}_{\mathbf{z},\mathbf{z}^{(\eta)}}^{(\eta)}\mathbf{F}_1^H\hat{\mathbf{R}}_{\mathbf{x},\mathbf{x}^{(\eta)}}\mathbf{F}_1^{(\eta)}\right)\right], \tag{69}$$

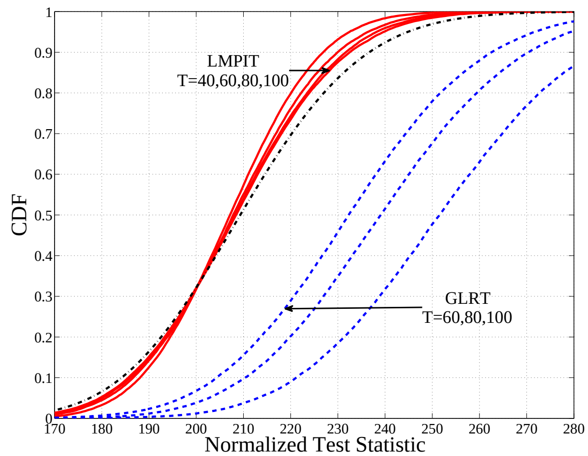

Fig. 1. Cumulative distribution function of the normalized statistics for testing $\mathcal{H}_{\mathbb{Q}}$ versus $\mathcal{H}_{\mathbb{C}^\eta}$. Hypothesis $\mathcal{H}_{\mathbb{Q}}$ and $n = 10$. Dash-dotted line: Wilks' asymptotic approximation. Red solid line: normalized LMPIT $T/2(\|\hat{\boldsymbol{\Phi}}_{\mathbb{R}^\eta}\|^2 - 4n)$. Blue dashed line: normalized GLRT statistic $2T\hat{\mathcal{P}}_{\mathbb{R}^\eta}$.

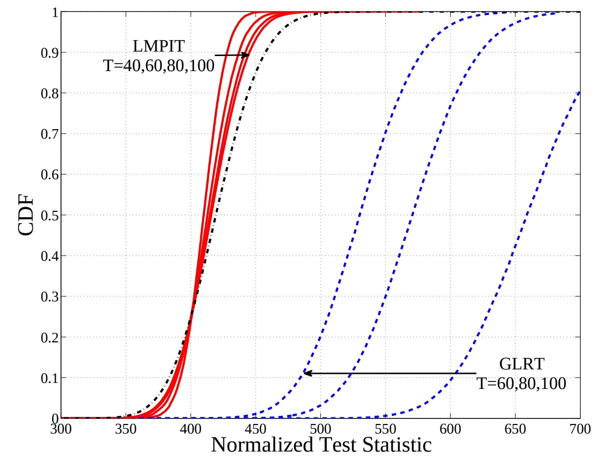

Fig. 2. Cumulative distribution function of the normalized statistics for the $\mathbb{C}^\eta$-properness test. Hypothesis $\mathcal{H}_{\mathbb{C}^\eta}$ and $n = 10$. Dash-dotted line: Wilks' asymptotic approximation. Red solid line: normalized LMPIT $T/2(\|\hat{\boldsymbol{\Phi}}_{\mathbb{C}^\eta}\|^2 - 4n)$. Blue dashed line: normalized GLRT statistic $2T\hat{\mathcal{P}}_{\mathbb{C}^\eta}$.

where now $\mathbf{R}_{\bar{\mathbf{z}},\bar{\mathbf{z}}} = \mathbf{R}_{\bar{\mathbf{x}},\bar{\mathbf{x}}}^{-1}(\mathcal{H}_{\mathbb{C}^\eta})$ and $\mathbf{R}_{\mathbf{z},\mathbf{z}} = \mathbf{I}_n$.

2) Approximation for $\mathbf{R}_{\bar{\mathbf{z}},\bar{\mathbf{z}}} \approx \mathbf{I}_{4n}$: The second step is significatively easier. It should be noted that now we only have one term ($\hat{\theta}_{\mathbb{C}^\eta}$ or $\hat{\theta}_{\mathbb{R}^\eta}$) instead of the three ($\hat{\theta}_{\eta,a}$, $\hat{\theta}_{\eta',a}$ and $\hat{\theta}_{\eta'',a}$) for the $\mathbb{Q}$-properness test. Therefore, we do not have cross-products of integrals, and it is easy to prove that the linear terms in the second-order Taylor's approximations vanish. Thus, the counterparts of eqs. (58) and (59) are

$$R_{\mathbb{C}^\eta} \propto \int_{\widetilde{\mathbf{F}}} \hat{\theta}^2_{\mathbb{C}^\eta} \left|\widetilde{\mathbf{F}}^H\widetilde{\mathbf{F}}\right|^T e^{-T\mathrm{Tr}\left(\widetilde{\mathbf{F}}^H\widetilde{\mathbf{F}}\right)} d\widetilde{\mathbf{F}}, \quad (70)$$

$$R_{\mathbb{R}^\eta} \propto \int_{\mathbf{F}_1} \hat{\theta}^2_{\mathbb{R}^\eta} \left|\mathbf{F}_1^H\mathbf{F}_1\right|^{2T} e^{-2T\mathrm{Tr}\left(\mathbf{F}_1^H\mathbf{F}_1\right)} d\mathbf{F}_1. \quad (71)$$

Furthermore, it is easy to see that in the scalar case, $\hat{\theta}_{\mathbb{R}^\eta}$ (and therefore also $R_{\mathbb{R}^\eta}$) is proportional to $\hat{\mathbf{R}}_{\mathbf{x},\mathbf{x}^{(\eta)}}$, which reduces to the sample canonical correlation (the absolute value of the sample correlation coefficient) between x and $x^{(\eta)}$. This means that the UMPIT for testing $\mathcal{H}_{\mathbb{Q}}$ versus $\mathcal{H}_{\mathbb{C}^\eta}$ coincides with the GLRT and LMPIT, and it reduces to the comparison of this sample canonical correlation with a threshold.

3) Solution of the Integrals: Finally, the solutions of the integrals in eqs. (70) and (71) follow the lines in Subsection V-C and, as expected, the final expressions for the two density ratios are

$$R_{\mathbb{C}^\eta} \propto \left\|\hat{\boldsymbol{\Phi}}_{\mathbb{C}^\eta}\right\|^2, \qquad R_{\mathbb{R}^\eta} \propto \left\|\hat{\boldsymbol{\Phi}}_{\mathbb{R}^\eta}\right\|^2. \quad (72)$$

VI. Numerical Examples

The performance of the proposed tests is illustrated in this section by means of some Monte Carlo simulations, which have been obtained with the help of the Matlab® quaternion Toolbox [68]. In particular, all the examples are based on T i.i.d. realizations of a zero-mean n-dimensional quaternion Gaussian vector with the appropriate SOS.

A. Null Distributions and Critical Regions

Analogously to the complex case [42], [43], the invariance of the testing problems under invertible conventional (or semi-widely) linear transformations can be directly exploited to obtain, by means of simulations with $\mathbf{R}_{\bar{\mathbf{x}},\bar{\mathbf{x}}} = \mathbf{I}_{4n}$, the distributions of the test statistics under the null (proper) hypothesis. As an example, Figs. 1-3 show the cumulative distribution functions (CDFs), for $n = 10$ and different sample sizes T, of the normalized LMPIT and GLRT statistics. The figures also show the approximated asymptotic distributions provided in eqs. (41)-(43), which corroborates the accuracy and the *conservative* nature of the Wilks' approach, i.e., for sufficiently large values of the test statistic, the Wilks' approximation can be seen as a lower bound for the actual distribution. From these figures, it is easy to obtain the thresholds $\gamma_{\mathbb{R}^\eta}$, $\gamma_{\mathbb{C}^\eta}$, $\gamma_{\mathbb{Q}}$ (or critical regions) for a fixed false alarm probability P_f. In particular, Tables IV-VI show the thresholds of the LMPITs for different values of n, T and P_f.

B. Receiver Operating Characteristic Curves (ROCs)

In order to compare the performance of the LMPITs and GLRTs, we consider an additional example with $n = 10$ and SOS as specified in Table VII. The ROC curves for different sample sizes (T) are shown in Figs. 4-6, where we can see that the LMPITs outperform their GLRT counterparts. Interestingly, although there is only a slight difference in the two first testing problems,[8] the difference is more noticeable in the $\mathbb{Q}$-properness test. Moreover, it is clear that the advantage of the LMPITs decreases with the sample size (T), which can be seen as a direct consequence of the assumption $T\hat{\theta}_a \approx 0$ made in the second-order Taylor's approximation in eq. (55) (equivalently in eqs. (70) and (71)). That is, as T (or the theoretical improperness) increases, the approximation in (55)

[8] A similar observation was made in [43] for the problem of testing the properness of a complex vector.

TABLE IV
CRITICAL VALUES $\gamma_{\mathbb{R}^\eta}$ FOR THE $\mathcal{H}_{\mathbb{Q}}$ VERSUS $\mathcal{H}_{\mathbb{C}^\eta}$ LMPIT.

n	P_f	$T=40$	$T=60$	$T=80$	$T=100$
1	0.1	4.3080	4.2065	4.1542	4.1249
	0.05	4.3796	4.2559	4.1922	4.1552
	0.01	4.5355	4.3663	4.2754	4.2243
2	0.1	8.7732	8.5217	8.3932	8.3149
	0.05	8.8757	8.5924	8.4475	8.3584
	0.01	9.0888	8.7468	8.5614	8.4497
4	0.1	18.2704	17.5346	17.1563	16.9303
	0.05	18.4284	17.6457	17.2430	17.0003
	0.01	18.7454	17.8670	17.4131	17.1395
6	0.1	28.5291	27.0619	26.3086	25.8553
	0.05	28.7362	27.2113	26.4273	25.9506
	0.01	29.1301	27.5054	26.6527	26.1365
8	0.1	39.5497	37.1062	35.8589	35.0987
	0.05	39.8015	37.2912	36.0023	35.2169
	0.01	40.2673	37.6435	36.2761	35.4492
10	0.1	51.3463	47.6636	45.7927	44.6524
	0.05	51.6236	47.8772	45.9633	44.7933
	0.01	52.1522	48.2831	46.2818	45.0590

TABLE V
CRITICAL VALUES $\gamma_{\mathbb{C}^\eta}$ FOR THE $\mathbb{C}^\eta$-PROPERNESS LMPIT.

n	P_f	$T=40$	$T=60$	$T=80$	$T=100$
1	0.1	4.5058	4.3431	4.2603	4.2085
	0.05	4.5869	4.4018	4.3045	4.2449
	0.01	4.7536	4.5204	4.4000	4.3226
2	0.1	9.3388	8.9106	8.6895	8.5546
	0.05	9.4540	8.9939	8.7539	8.6079
	0.01	9.6875	9.1617	8.8882	8.7167
4	0.1	20.1061	18.7985	18.1243	17.7085
	0.05	20.2779	18.9313	18.2295	17.7967
	0.01	20.6177	19.1887	18.4322	17.9697
6	0.1	32.3714	29.7046	28.3250	27.4872
	0.05	32.5941	29.8738	28.4656	27.6032
	0.01	33.0104	30.2059	28.7307	27.8339
8	0.1	46.1358	41.6334	39.3039	37.8826
	0.05	46.3818	41.8480	39.4823	38.0293
	0.01	46.8472	42.2286	39.8055	38.3108
10	0.1	61.3873	54.5712	51.0530	48.9045
	0.05	61.6485	54.8021	51.2510	49.0720
	0.01	62.1150	55.2275	51.6185	49.3896

TABLE VI
CRITICAL VALUES $\gamma_{\mathbb{Q}}$ FOR THE $\mathbb{Q}$-PROPERNESS LMPIT.

n	P_f	$T=40$	$T=60$	$T=80$	$T=100$
1	0.1	4.7193	4.4832	4.3621	4.2909
	0.05	4.8263	4.5562	4.4172	4.3341
	0.01	5.0653	4.7186	4.5358	4.4284
2	0.1	9.9599	9.3208	8.9937	8.7971
	0.05	10.1269	9.4307	9.0782	8.8670
	0.01	10.4571	9.6574	9.2514	9.0066
4	0.1	22.1940	20.1686	19.1415	18.5154
	0.05	22.4625	20.3561	19.2855	18.6327
	0.01	22.9937	20.7132	19.5709	18.8641
6	0.1	36.7460	32.5787	30.4612	29.1841
	0.05	37.1098	32.8305	30.6547	29.3452
	0.01	37.8063	33.3194	31.0351	29.6567
8	0.1	53.6284	46.5538	42.9674	40.8030
	0.05	54.0729	46.8749	43.2157	41.0089
	0.01	54.8952	47.4747	43.6898	41.4068
10	0.1	72.8242	62.1021	56.6590	53.3605
	0.05	73.3321	62.4772	56.9543	53.6069
	0.01	74.2717	63.1714	57.5134	54.0752

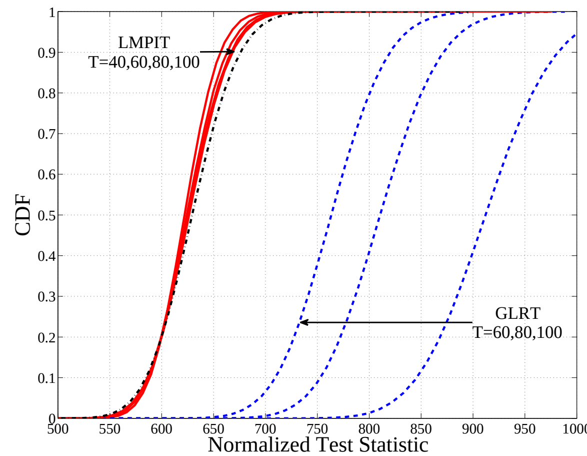

Fig. 3. Cumulative distribution function of the normalized statistics for the $\mathbb{Q}$-properness test. Hypothesis $\mathcal{H}_{\mathbb{Q}}$ and $n=10$. Dash-dotted line: Wilks' asymptotic approximation. Red solid line: normalized LMPIT $T/2(\|\hat{\mathbf{\Phi}}_{\mathbb{Q}}\|^2-4n)$. Blue dashed line: normalized GLRT statistic $2T\hat{\mathcal{P}}_{\mathbb{Q}}$.

TABLE VII
SECOND-ORDER STATISTICS FOR THE SIMULATION EXAMPLES

	$\mathbf{R}_{\mathbf{x},\mathbf{x}}$	$\mathbf{R}_{\mathbf{x},\mathbf{x}^{(\eta)}}$	$\mathbf{R}_{\mathbf{x},\mathbf{x}^{(\eta')}}$	$\mathbf{R}_{\mathbf{x},\mathbf{x}^{(\eta'')}}$
$\mathcal{H}_{\mathcal{I}}$	$\mathbf{I}_{10}$	$0.2\mathbf{I}_{10}$	$0.2\mathbf{I}_{10}$	$0.2\mathbf{I}_{10}$
$\mathcal{H}_{\mathbb{C}^\eta}$	$\mathbf{I}_{10}$	$0.2\mathbf{I}_{10}$	$\mathbf{0}_{10\times 10}$	$\mathbf{0}_{10\times 10}$
$\mathcal{H}_{\mathbb{Q}}$	$\mathbf{I}_{10}$	$\mathbf{0}_{10\times 10}$	$\mathbf{0}_{10\times 10}$	$\mathbf{0}_{10\times 10}$

becomes less accurate, and the GLRTs could even outperform the LMPITs.

VII. CONCLUSIONS

This paper has addressed the problem of testing the properness of a quaternion random vector. Focusing on the two main kinds of quaternion properness, three different binary hypothesis testing problems have been considered, and the

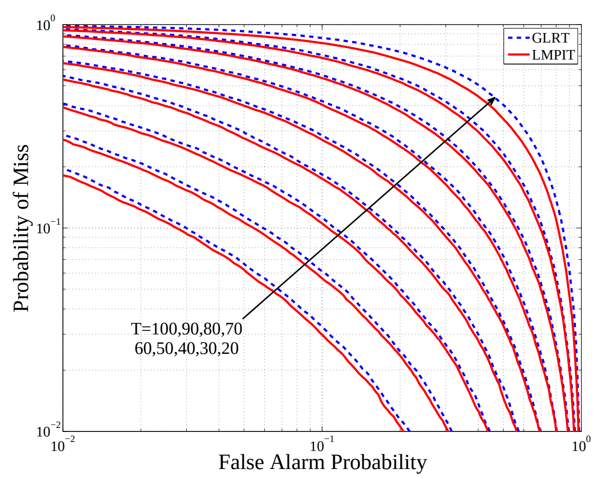

Fig. 4. Receiver Operating Characteristic. $\mathcal{H}_{\mathbb{Q}}$ versus $\mathcal{H}_{\mathbb{C}^\eta}$ tests. $n=10$ and SOS as specified in Table VII.

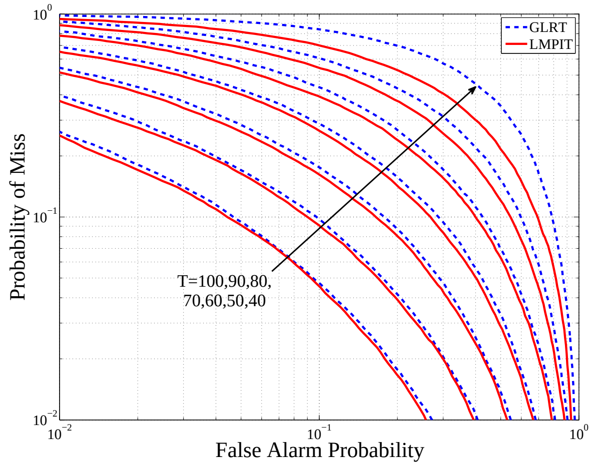

Fig. 5. Receiver Operating Characteristic. $\mathbb{C}^{\eta}$-properness tests. $n = 10$ and SOS as specified in Table VII.

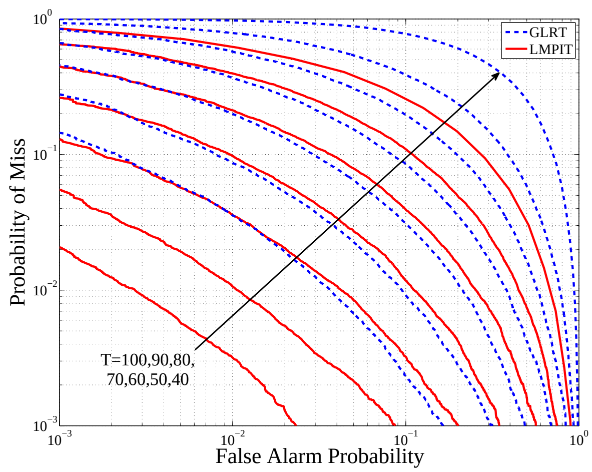

Fig. 6. Receiver Operating Characteristic. $\mathbb{Q}$-properness tests. $n = 10$ and SOS as specified in Table VII.

respective locally most powerful invariant tests (LMPITs) have been presented. Interestingly, even though there is not a simple expression for the maximal invariant of the most complicated testing problem ($\mathbb{Q}$-properness test), the LMPITs can be obtained thanks to the Wijsman's theorem. The proposed tests result in simple detection rules based on the Frobenius norm of three previously defined coherence matrices. Furthermore, we have analyzed the connections with the generalized likelihood ratio tests (GLRTs), with the problem of testing for the properness of a complex vector, and with the sphericity tests for four-dimensional real (or two-dimensional proper complex) vectors. Finally, some numerical examples have shown that the LMPITs generally outperform their GLRT counterparts, and the performance gain is especially noticeable for small sample sizes and the $\mathbb{Q}$-properness test.

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