Design of Simultaneous Sampling Systems Based on Fractional Delay Lagrange Filters

David Luengo, Carlos J. Pantaleón, Jesús Ibáñez, and Ignacio Santamaría

***Abstract*—Digital filtering is a common approach to achieve simultaneous sampling of several input signals acquired with a multiplexing delay. In this brief, an error bound is obtained for Lagrange interpolation filters as a function of the oversampling ratio of the input signals, the fractional delay, and the filter's order. This bound can be used to ensure that the error is small enough to maintain a desired resolution (number of significant bits), thus leading to design equations for simultaneous sampling systems. For example, using these equations, we are able to find that an oversampling ratio of 71 is necessary to maintain a resolution of 12 bits with a first order Lagrange's filter, while a sixth-order filter is required when the oversampling ratio is only five. The theoretical results are validated through simulation, and the computational cost of the Lagrange's interpolator is compared with a polyphase filter.**

***Index Terms*—Analog–digital conversion, data acquisition, digital filters, Lagrange interpolation, signal sampling, time delay.**

I. Introduction

In many signal processing applications, it is required to delay a signal by a fractional multiple of the sampling period: irrational sampling-rate conversion [1], timing adjustment, and synchronization in digital modems [2], [3], high-resolution pitch prediction [4], high-precision beam steering [5], discrete-time modeling of acoustic tubes [6], or multichannel acquisition systems [7]. A typical low-cost data acquisition system for multiple input channels consists of an analog multiplexer followed by a single sample-and-hold and an analog to digital converter. This architecture introduces a fractional sample delay between consecutive channels, which must be corrected in applications that require a high resolution. Besides the obvious, but costly, solution of using one sample-and-hold for each channel, two alternative solutions exist: multirate techniques and digital filtering.

Multirate techniques [8] solve this problem by interpolating the incoming signals by a factor M (M being the number of channels), then delaying every signal by an integer number of samples at the resulting (higher rate) sampling period, and finally decimating to restore the original sampling rate. This method can be efficiently implemented using polyphase structures [8]. An alternative approach is to use low-order fractional delay filters (FDF's), for which an extensive description of design techniques is provided in [9]. These design techniques usually attempt to achieve a good fractional delay over the full band. However, in data acquisition systems, the input signals are typically highly oversampled to facilitate the design of the analog antialiasing filters. Therefore, only the low portion of the spectrum contains useful information. In this case, Lagrange interpolators are the most attractive solution because of their maximally flat frequency response and smooth transition approximation to the ideal all-pass filter at low frequencies with very few taps [9].

The problem addressed in this brief is the analysis of fractional delay Lagrange filtering compensation of multiplexing delay. If the resolution (number of significant bits) of the A/D converter is maintained for every input signal, then simultaneous sampling is achieved. In Section II, the problem is stated, the worst error case presented, and an error bound is obtained that relates the number of significant bits of the A/D converter with the Lagrange filter's order and the oversampling ratio. It is shown that by increasing the oversampling ratio, the distortion introduced by the delay compensation stage can be made as low as desired, and expressions are developed that, given a filter's order, provide the oversampling required to achieve a certain resolution. In Section III, the results of the simulations are shown and the computational cost of the proposed solution is compared with a polyphase filter.

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II. Fractional Delay Compensation

A. Problem Statement

Let us consider an analog input signal $x(t)$ bandlimited in the range $[0, f_o]$ Hz (minimum period $T_o = 1/f_o$), sampled at $f_s > 2f_o$ Hz (oversampling ratio $R = f_s/2f_o = T_o/2T_s$) using an analog-to-digital converter (ADC) with a certain resolution. This input signal must be delayed by a desired amount D which, in general, will consist of an integer number of samples and a fractional delay. This delay will be accomplished using an Nth-order Lagrange filter ($N + 1$ taps), leading to a total delay

$$D = \frac{N}{2} + d \tag{1}$$

where $N/2$ is the delay of a causal linear phase FIR filter, which will already have a fractional part if N is odd, and d is the additional fractional delay. This delay will be in the range $[-0.5, 0.5]$ to obtain an interpolation region around the filter's center of symmetry $(D = N/2)$. Restricting the interpolation range to this region (between the two central taps for N odd, and centered on the filter's central tap for N even) will produce the best interpolation results for a given set of samples [9].

Our goal is to develop expressions that relate the interpolation error with the oversampling ratio R, and the filter's order N. Then, if the interpolation error does not exceed the maximum level of the quantization noise imposed by the resolution of the ADC, it will be masked by it, and the ADC's resolution will be preserved. Therefore, we will be able to ensure that the signal is correctly delayed without losing signal resolution.

B. Interpolation Error Upper Bound

To guarantee that the ADC's resolution is maintained for a given oversampling ratio and filter's order, we must find the conditions under which the worst interpolation error occurs. Hence, we will be able to obtain an upper error bound, and develop useful design equations. The interpolation error is simply given by the difference between the signal interpolated using the ideal all-pass filter xi and the signal interpolated using Lagrange's FDF x_r

$$e[n] = x_i[n-D] - x_r[n-D] = x[n] * (h_i[n] - h_L[n]) \tag{2}$$

where $*$ stands for the convolution operator. The ideal filter is an all-pass filter that simply introduces a delay D. Therefore, its impulse response will be a delayed sinc

$$h_i[n] = \frac{\sin[\pi(n-D)]}{\pi(n-D)} = \sin c[n-D] \tag{3}$$

and the coefficients of the Lagrange's FDF are given by [10]

$$h_L[n] = \prod_{\substack{i=0 \\ i\neq n}}^{N} \frac{D-i}{n-i} = (-1)^{N-n}\binom{D}{n}\binom{D-n-1}{N-n}. \tag{4}$$

TABLE I
LAGRANGE FRACTIONAL DELAY FILTER COEFFICIENTS

N	h[0]	h[1]	h[2]	h[3]
1	1-*D*	*D*		
2	(*D*-1)(*D*-2)/2	-*D*(*D*-2)	*D*(*D*-1)/2	
3	-(*D*-1)(*D*-2)(*D*-3)/6	*D*(*D*-2)(*D*-3)/2	-*D*(*D*-1)(*D*-3)/2	*D*(*D*-1)(*D*-2)/6

In Table I, the coefficients for filters of orders from one (linear interpolation) to three are shown as a function of D. The symmetry of Lagrange's FDF coefficients ($h[N-n]$ for $D = N/2 - d$ is identical to $h[n]$ for $D = N/2 + d$) causes the error to be identical for d and $-d$. Consequently, we will consider d only in the range [0, 0.5].

The interpolation error depends on two factors: the input signal and the difference between the ideal and actual filters, which depends on the fractional delay and the frequency (or equivalently the oversampling ratio). The worst error case is a sinusoidal input signal whose frequency is the maximum allowed input frequency [1], $x(t) = A\cos(2\pi f_o t)$. The worst case of fractional delay will be the one for which the distance from the interpolation point to the nearest sample is maximum [9]: $d = 0$ for N odd, and $d = 0.5$ for N even. However, we will keep D as a parameter in subsequent expressions. The interpolation error in this case is

$$e[n] = A\left\{\cos[(n-D)\omega_o] - \sum_{k=0}^{N} h_L[k]\cos[(n-k)\omega_o]\right\}. \quad (5)$$

We have a LTI system, so obviously, the error may be expressed as a sinusoidal signal of the same frequency as the input with a certain amplitude and phase

$$e[n] = A_e\cos[n\omega_o + \phi_e]. \quad (6)$$

In (6), the phase is irrelevant for our purpose of obtaining an upper bound for the error, which may be obtained straightforwardly by taking the absolute value of the well-known Lagrange's remainder formula [11]

$$|e[n]| = \frac{1}{(N+1)!}\left|\frac{\partial^{N+1}x(t)}{\partial t^{N+1}}\right|_{t=\xi(t)} \prod_{k=0}^{N}|t - t_i| \quad (7)$$

where $\xi(t)$ is an unspecified instant belonging to the interpolation interval, in this case $|t - t_i| = |D - i|T_s$, and the absolute value of the $(N+1)$-th derivative for a sinusoidal input is always lower than or equal to $(2\pi f_o)^{N+1}$. Therefore, an upper bound for (7) may be written as

$$|e[n]| \le \frac{A}{(N+1)!}\left(\frac{2\pi f_o}{f_s}\right)^{N+1}\prod_{k=0}^{N}|D-i|. \quad (8)$$

Now, rearranging terms, we may obtain an estimate of A_e as

$$A_e \cong A\left(\frac{\pi}{RC_N(D)}\right)^{N+1} \quad (9)$$

where $C_N(D)$ is a function that does not depend on the oversampling ratio, and whose value is given by

$$C_N(D) = \left((N+1)!/\prod_{i=0}^{N}|D-i|\right)^{1/N+1}. \quad (10)$$

Relating now the error's amplitude to the ADC's resolution will lead us to the design equations we search in the next section.

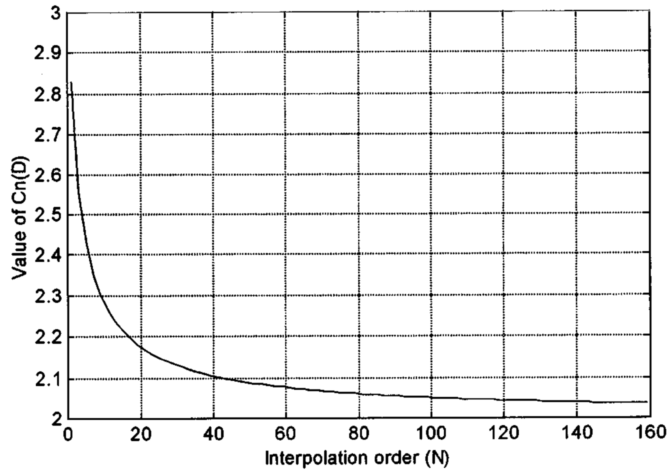

Fig. 1. $C_n(D)$ for a total fractional delay 0.5 as a function of the filter's order.

C. Design Equations

Let us consider a bipolar ADC with $B+1$ significant bits and a range $[-X_m, X_m]$ or, equivalently, a unipolar ADC with B significant bits and a range $[0, X_m]$. In order to consider that the fractional delay compensation stage introduces no additional distortion, the maximum interpolation error should be less than the error introduced by the ADC in the quantization stage, which is usually expressed in terms of the least significant bit (LSB). Thus, to achieve real simultaneous sampling with a FDF, the amplitude of the error must be

$$A_e \le \frac{\eta X_m}{2^B} \quad (11)$$

where η is a number that will generally be in the range [0.5, 1], being 0.5 for an ideal ADC, and 1 meaning that a resolution bit is lost during the quantization stage. If the signal uses the full range of the ADC ($A = X_m$ for a bipolar ADC, and $A = X_m/2$ for a unipolar ADC), and we consider an ideal bipolar ADC $(\eta = 0.5)$, we may immediately obtain the value of the oversampling ratio necessary to achieve true simultaneous sampling using Lagrange interpolation

$$R \ge \frac{\pi}{C_N(D)}2^{(B+1)/(N+1)}. \quad (12)$$

Conversely, we may obtain the resolution ($B+1$ bits) attained using a certain oversampling ratio

$$B+1 \cong (N+1)\log_2\left(\frac{RC_N(D)}{\pi}\right). \quad (13)$$

Obtaining an expression for the required filter's order for a certain oversampling ratio and resolution is more complex because of the dependence of $C_N(D)$ on the filter's order. However, plotting this function versus N for the worst error case ($d = 0$ for N odd and $d = 0.5$ for N even), we observe in Fig. 1 that the maximum of $C_N(D)$ is $\sqrt{8}$ obtained for $N = 1$, and $C_N(D)$ seems to tend to 2 as N increases.

Then, as the usual design scenario involves the design for the worst case, these two values may be used as higher and lower bounds to estimate N

$$\frac{B+1}{\log_2\left(\sqrt{8}R/\pi\right)} \le N+1 \le \frac{B+1}{\log_2(2R/\pi)}. \quad (14)$$

These upper and lower bounds, which in general will be non integer numbers, may be used to obtain the actual filter's order required iteratively (i.e., initialize N as the lowest integer greater or equal than the left side of the inequality, calculate the resolution obtained using (13) and continue iterating until the desired requirements are fulfilled). Equations (12)–(14) are also valid for a unipolar ADC substituting $(B+1)$ by B.

III. Simulation Results

To validate the previous expressions through simulation, we have generated a discrete time sinusoidal signal $x[n]$ with random phase and amplitude $A = 1$. This signal has been filtered using Lagrange filters of orders from 0–6 (with $d = 0$ for odd filters' orders and $d = 0.5$ for even orders), and compared to the ideally delayed replica $x[n-D]$. The oversampling ratio was changed from 1 to 100 with step one. The simulation results, as well as the theoretical solution obtained from (13), are shown in Fig. 2(a). The upper bound obtained is such an accurate approximation for moderate and high oversampling ratios that both sets of curves cannot be distinguished. The curves show how with a linear interpolator and an oversampling ratio as high as 100, hardly 13 bits of resolution are maintained, while using a sixth-order filter and $R = 10$ a resolution of 20 bits is possible.

In Fig. 2(b)the range of validity of the approximation is studied by performing a zoom of the low oversampling ratio region for $N = 1$ to 5. Now R is changed between 1.5 and 3.5, with step 0.01. As the design equations offer an upper error bound, the error is overestimated and, therefore, the theoretical curves (dashed line) underestimate the true resolution achieved (solid line).

Finally, the computational cost of the Lagrange's FDF is compared with a multirate correction scheme implemented using a polyphase filter with M branches (M being the number of channels). This filter has been designed using the Parks McClellan equiripple design technique with a passband $[0, \pi/MR]$, a stopband $[(2\pi - \pi/R)/M, 2\pi]$, an in-band ripple $r_p = 2^{-(B+1)}$, and an out-of-band attenuation of $A = 6.02(B+1) + 1.76$ dB. Table II shows the number of taps of the Lagrange's FDF required for six values of R and five different resolutions, compared to the number of taps of the corresponding branch of the polyphase filter required to perform the delay (in parenthesis). The values of the polyphase filter correspond to $M = 16$. For lower values of M, the number of taps per branch of the polyphase filter may be slightly lower, while for greater values of M, the number of taps per channel may be slightly greater, but, in general, remains fairly constant. The computational cost of Lagrange's FDF is found to be lower than that of the polyphase alternative for oversampling ratios greater than ten.

IV. Conclusion

Simultaneous sampling of several input channels can be achieved using Lagrange's fractional delay filters. The performance of this approach depends on the oversampling ratio of the input signals and the filter's order. In applications where a high oversampling ratio is typical, such as multichannel data acquisition systems, Lagrange FDF's are very advantageous because of their smoothness at low frequencies and low order required to obtain a good approximation to the ideal filter.

In this brief, we have obtained a bound for the interpolation error of Lagrange's fractional delay filters. Based on this bound, we have developed expressions that relate the number of significant bits with the oversampling ratio, the filter's order, and the fractional delay. It is shown that, by increasing the filter's order for a given oversampling ratio, or conversely, increasing the oversampling ratio for a given filter's order, any desired resolution may be achieved. These theoretical results are validated through simulation. When the oversampling ratio is greater than five, the interpolation errors obtained by simulation closely resemble the expected theoretical ones. Finally, the computational cost of the method is compared to the alternative of polyphase filters and found to be lower for oversampling ratios greater than ten.

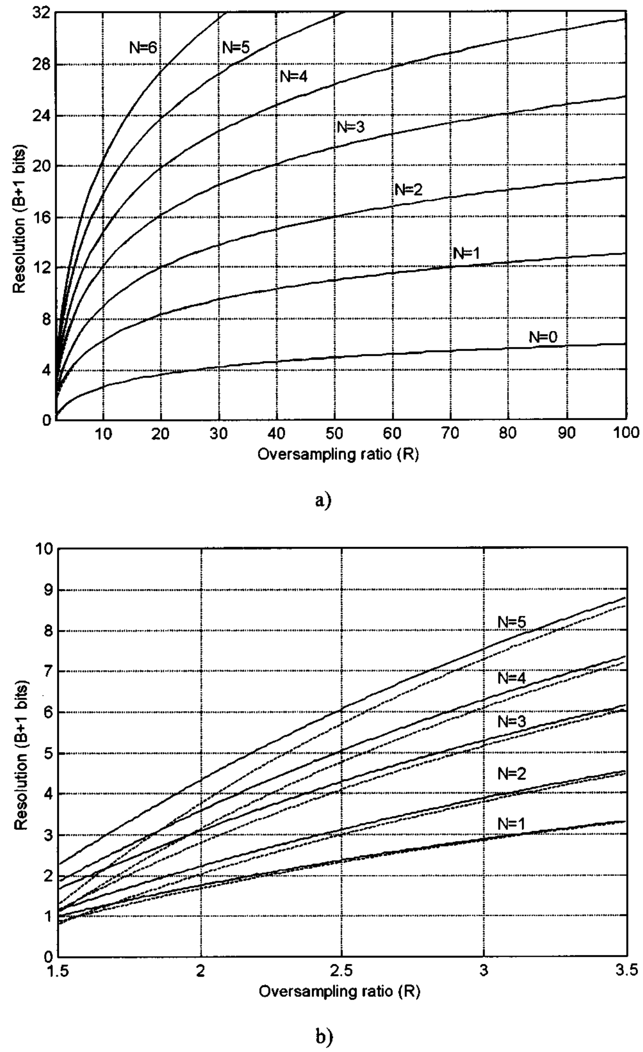

Fig. 2. Number of significant bits (theoretical in dashed line and simulated in solid line) for several filters' orders as a function of the oversampling ratio (R) for: (a) R in the range [1, 100] and (b) low oversampling ratios (R in the range [1.5, 3.5]).

TABLE II
COMPARISON OF THE NUMBER OF TAPS REQUIRED TO OBTAIN A DESIRED RESOLUTION OF LAGRANGE'S FDF AND POLYPHASE FILTER FOR $M = 16$ (IN PARENTHESIS)

R	Resolution (B+1 bits)				
	8	12	16	24	32
2	17 (6)	27 (9)	38 (12)	60 (18)	82 (24)
5	4 (4)	7 (6)	9 (8)	14 (12)	18 (15)
10	3 (3)	4 (5)	6 (7)	9 (10)	12 (13)
25	2 (3)	3 (5)	4 (6)	6 (10)	8 (12)
50	2 (3)	3 (5)	4 (6)	5 (10)	7 (12)
100	2 (3)	2 (5)	3 (6)	4 (9)	6 (12)

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On Parallelism in the Ensemble Sense Between Time-Series Models and Discrete Wavelet Transforms of Stochastic Signals

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***Abstract*—The brief is concerned with wavelet transforms (WT) of colored (correlated) discrete stochastic signals (time-series) and their relation to AR/ARMA models of the same signals. It derives the relations between AR/ARMA models of WT coefficients and AR/ARMA model of the signal, which eliminates the need to actually perform the WT of such signals in order to derive models of WT coefficients. The brief explains how to arrive at the WT coefficient ARMA models from the signal's ARMA model and vice-versa to show that WT properties of the ensemble are fully predictable from the signal's AR/ARMA model. In particular, the authors have shown that from AR/ARMA parameters of the stochastic signal alone, one can derive a realization of the WT coefficients of that stochastic signal and that by invoking the inverse WT on those coefficients, one then retrieves a stochastic signal whose AR/ARMA structure is the same as that of the original signal. It is noted that for a stochastic signal, signal parameters, rather than a particular realization, convey the information on the signal.**

***Index Terms*—ARMA models, stochastic models, stochastic signals, time series, wavelets.**

I. INTRODUCTION

The present brief is concerned with interrelating stochastic process models with wavelet transform (WT) models. In particular it investigates the models of WT coefficients as obtained from ARMA models of colored noise signals. It also investigates ARMA models of colored noise signals as obtained from models of WT coefficients of such signals and as obtained directly from the signal itself. In recent years, several authors were concerned with the relations between WT and stochastic processes. Basseville *et al.* [1] introduce a notion of multiresolution stochastic processes on n-ary trees and their stationarity. The work in [1] was extended in [2], and later in [3]. WT of stochastic processes (fractional Brownian motion) has also been discussed in [4] and [5]. Another brief by Dijkerman and Mazumdar [6] proposes multiresolution stochastic models based on the WT coefficients. The present work (similarly to [6]) discusses the AR/ARMA models of WT coefficients of a wide sense stationary stochastic time process. In contrast to [6], the present brief concentrates on stochastic models defined on the WT coefficients (of a stochastic process) on the same scale of the WT. The brief derives the models of WT coefficients from the estimated model of the original stochastic process rather than attempting to estimate these models from the WT coefficients directly. This allows the present derivation to overcome the reliability problem associated with estimation based on very few coefficients. Given a constant number of samples of the original stochastic process (used in the estimation of the AR/ARMA model of the process itself and subsequently, models of its WT coefficients), the number of WT coefficients (potentially used to estimate a stochastic model of those coefficients) decreases exponentially with scale. In this brief, we are mostly concerned with ARMA models since they represent a more general case and since models generated by the algorithm described below are or become ARMA models. In the case of stable and invertible models, AR and ARMA are fully

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