Optimal Estimation of Chaotic Signals Generated by Piecewise-Linear Maps

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***Abstract*—Chaotic signals generated by iterating piecewise-linear (PWL) maps on the unit interval are highly atractive in a wide range of signal processing applications. In this letter, optimal estimation algorithms for signals generated by iterating PWL maps and observed in white noise are derived based on the method of maximum likelihood (ML). It is shown how the phase space of the map may be decomposed into a number of regions and how the estimation problem is linear in each of these regions. The final ML estimate is obtained as the best performing of these "local" solutions.**

***Index Terms*—Chaos, maximum likelihood estimation.**

I. Introduction

CHAOTIC signals, signals generated by a nonlinear dynamical systems in a chaotic state, have received much attention in the past few years. Several authors have proposed signal estimation algorithms [1]–[4], but most methods are suboptimal, and maximum likelihood (ML) estimators have only been developed for the tent map dynamics [3]. In this letter, we develop ML estimators for the general class of piecewise-linear (PWL) maps on the unit interval. We apply the symbolic dynamics theory to finite-length chaotic sequences and show how the error surface has a finite number of minima, each one associated to each admissible symbolic sequence. Suitable estimators are derived for each of these minima. The final ML estimate is given by the best performing of the previous "local" estimates.

II. Symbolic Dynamics of Piecewise-Linear Maps

In this letter, we consider piecewise-linear maps F: $[0, 1] \rightarrow [0, 1]$. The interval $[0, 1]$ is partitioned into disjoint convex intervals E_i, $i \in L$, where L is an index set. Then F is defined as

$$F(x) = a_i x + b_i \quad \text{if } x \in E_i \tag{1}$$

where all the a_i and b_i are known constants. This formulation includes all piecewise linear maps on the unit interval, continuous or not, including, for example, Markov maps. The extension to maps in higher dimensions is quite straightforward [5]. We will denote F^k the k-fold composition of F. Chaotic signals may be generated by iterating an unknown initial condition $x[0] \in [0, 1]$ according to

$$x[n] = F(x[n-1]). \tag{2}$$

The index set L is an alphabet consisting of a set of symbols. We will denote the symbolic sequence (sometimes called itinerary) associated to a chaotic signal of length $N+1$ as a length N sequence of symbols $\boldsymbol{s} = s[0], s[1], \cdots, s[N-1]$, where $s[k] = i$ if $F^k(x[0]) \in E_i$. We will define S_N as the map that maps an initial condition in $[0, 1]$ to its corresponding symbolic sequence of length N.

We can define another partition of the phase space in a collection of P intervals R_j composed of the points in $[0, 1]$ that share a common length N symbolic sequence s_j, which is given a certain itinerary $\boldsymbol{s}_j$. We define $R_j = \{x \in [0, 1]: S_N(x) = \boldsymbol{s}_j\}$. Every point in the phase space belongs to one and only one of these sets, and, if the $E_i's$ are convex sets, the $R_j's$ are also convex [5]. This partition is also known as natural partition [6]. These regions can be obtained by iterating backward from the whole phase space with each symbolic sequence and the regions limits can be obtained by iterating backward from the breakpoints of $F(x)$ [6]. If all the linear components are onto, all the symbolic sequences are admissible, and there are $P = M^N$ regions R_j, where M denotes the length of the alphabet set L. If any of the linear components is not onto, some symbolic sequences will not be admissible and $P < M^N$.

We will denote $F_{\boldsymbol{s}}^k(x[0])$ as the k-fold composition of F for an initial condition $x[0]$ with $S_N(x[0]) = \boldsymbol{s}$. Given a known itinerary, we can write a closed form expression for $F_{\boldsymbol{s}}^k(x[0])$. If we define

$$A_n^k = \left(\prod_{j=k-n}^{k-1} a_{s[j]} \right) \tag{3}$$

and $A_0^k = 1$, then we have

$$F_{\boldsymbol{s}}^k(x[0]) = A_k^k x[0] + \sum_{n=0}^{k-1} A_n^k b_{s[k-1-n]}. \tag{4}$$

Note the linear dependence on the initial condition in (4). The index $\boldsymbol{s}$ stresses the fact that (4) is only equivalent to $F^k(x[0])$ if the itinerary of $x[0]$ is given by $\boldsymbol{s}$.

As a conclusion, given any PWL map, we can define a partition of the phase space in convex intervals R_j in which all the points share the same symbolic sequence of length N. In this

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case, given an itinerary $\boldsymbol{s}_j$, (4) is a closed form expression for $F^k(x)$ in the domain R_j.

III. ML Estimation of PWL Maps

The data model for the problem we are considering is

$$y[n] = x[n] + w[n], \quad n = 0, 1, \cdots, N \tag{5}$$

where $w[n]$ is a stationary, zero-mean white Gaussian noise with variance σ^2, and $x[n]$ is generated by iterating some unknown $x[0] \in [0, 1]$ according to (2) using (1).

Since the observation vector is a collection of Gaussian independent random variables, ML signal estimation produces the initial condition that minimizes

$$J(x[0]) = \sum_{k=0}^{N} \left(y[k] - F^k(x[0])\right)^2. \tag{6}$$

Due to the invariance property of ML estimators, the whole signal may be estimated from the ML estimate of $x[0]$, so we will consider only the ML estimation of the initial condition. Let us define an indicator (sometimes called characteristic) function

$$\chi_j(x) = \begin{cases} 1, & \text{if } x \in R_j \\ 0, & \text{if } x \notin R_j \end{cases} \tag{7}$$

and using (4), we can express (6) in a certain region R_j

$$J_j(x[0]) = \sum_{k=0}^{N} \left(y[k] - F^k_{\boldsymbol{s}_j}(x[0])\right)^2. \tag{8}$$

So we can write (6) as

$$J(x[0]) = \sum_{j=1}^{P} \chi_j(x[0]) J_j(x[0]). \tag{9}$$

It is easy to realize, due to the linear dependence on the initial condition of (4), that $J_j(x[0])$ is a quadratic function in each R_j. Differentiating and solving for the unique minimum, we obtain

$$\hat{x}_j[0] = \frac{\sum_{k=0}^{N} \left(y[k] - \sum_{n=0}^{k-1} A_n^k b_{s_j[k-1-n]} \right) A_k^k}{\sum_{k=0}^{N} \left(A_k^k\right)^2} \tag{10}$$

where A_n^k is given by (3) with itinerary $\boldsymbol{s}_j$, and $s_j[n]$ is the nth component of $\boldsymbol{s}_j$.

This is the ML estimate of $x[0]$ with a known itinerary if $\hat{x}_j[0] \in R_j$. Otherwise, the minimum of the quadratic error surface is outside the admissible range, that is, $\hat{x}_j[0]$ does not have an itinerary given by $\boldsymbol{s}_j$. The minimum of $J(x[0])$ in the R_j interval is produced by the closest value to $\hat{x}_j[0]$ in R_j. If we denote $x^j_{\min}$, $x^j_{\max}$, the minimum and maximum of R_j, the ML estimate associated with itinerary $\boldsymbol{s}_j$ is

$$\hat{x}^j_{\mathrm{ML}}[0] = \begin{cases} \hat{x}_j[0], & \hat{x}_j[0] \in R_j \\ x^j_{\min}, & \hat{x}_j[0] < x^j_{\min} \\ x^j_{\max}, & \hat{x}_j[0] > x^j_{\max} \end{cases} \tag{11}$$

TABLE I
MSE in dB for the ML Estimator and the Suboptimal HC-ML

$a = 0.95, N = 7$		
SNR	ML	HC-ML
10 dB	25.0	24.0
15 dB	30.7	28.6
20 dB	36.7	33.4
25 dB	42.7	38.0
30 dB	48.7	43.3
35 dB	54.6	50.0
40 dB	61.5	57.0
45 dB	69.1	65.3
50 dB	75.2	72.5

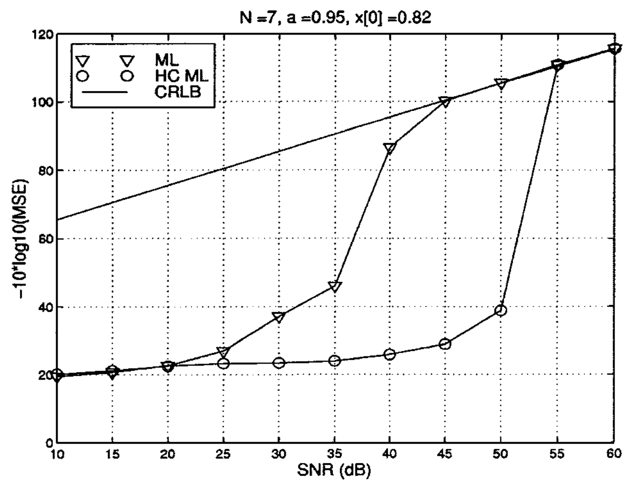

Fig. 1. Mean square error (MSE) for the ML estimator and the suboptimal HC-ML.

and finally, the ML estimate of $x[0]$ is given by the $\hat{x}^j_{\mathrm{ML}}[0]$ that minimizes (6). If we define

$$k = \arg\min_j \left(J\left(\hat{x}^j_{\mathrm{ML}}[0]\right)\right), \quad j = 1, 2, \cdots, P \tag{12}$$

then $\hat{x}_{\mathrm{ML}}[0] = \hat{x}^k_{\mathrm{ML}}[0]$.

In the general case, ML estimation demands the computation of a maximum of M^N estimates (each one associated with an admissible itinerary) and the selection of the one that minimizes (6). In some cases, it is possible to develop more efficient approaches. That is the case with tent maps, where an efficient recursive ML estimation algorithm has been proposed [3].

IV. Simulation Results

In this section, we analyze the performance of the ML estimator for the following PWL mapping:

$$F(x) = \begin{cases} x/a, & 0 \le x \le a \\ (1-x)/(1-a), & a < x \le 1 \end{cases} \tag{13}$$

which is called a skew-tent map with parameter a in the range [0, 1] [7]. When $a = 0.5$, F becomes the tent map. We compare the ML estimate with a suboptimal approach we call hard censoring ML (HC-ML). The HC-ML estimates the itinerary directly by hard censoring the noisy sequence and then applying (10) and (11) to obtain the estimate. This approach reduces the computational cost. It is in the direction of those proposed in [1], [2], [4], and it can be considered of similar performance. We simulate a skew-tent map with $a = 0.95$ and generate length $N+1$ registers with $N = 7$. We select 999 initial conditions equally spaced in the range [0.01, 0.999]. Table I shows the mean square error (MSE) obtained by Monte Carlo simulations, averaging 1000 trials for each initial condition and SNR. Fig. 1 shows the results for an initial condition of $x[0] = 0.82$. The ML estimate attains the Cramer–Rao lower bound (CRLB) for SNR's over 45 dB, while the HC-ML needs 55 dB.

V. CONCLUSIONS

In this letter, we have developed the ML estimator for the initial condition of chaotic signals generated by any piecewise-linear map on the unit interval. We have shown that the problem may be decomposed in the computation of a maximum of M^N (where M is the number of linear components in the map and $N+1$ the number of data samples) estimates, one for each admissible itinerary. For each fixed itinerary, the error surface is quadratic, with a unique minimum. If the minimum falls inside the region R_j associated with the given itinerary, then it is the local ML estimate. If the minimum does not belong to R_j, then the closest value to that minimum in the domain R_j is the local ML estimate. The final ML estimate is the best performing among these previous local estimates.

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