EXACT CLOSED-FORM EXPRESSIONS FOR THE SUM CAPACITY AND INDIVIDUAL USERS' RATES IN BROADCAST ERGODIC RAYLEIGH FADING CHANNELS

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ABSTRACT

We derive closed-form expressions for the sum capacity of broadcast ergodic Rayleigh fading channels considering two power allocation policies along time: constant transmit power and optimal power allocation. In addition, we derive closed-form expressions for the individual users' rates when the sum capacity is achieved. We use these expressions to obtain the sum capacity and individual rates in a variety of broadcast channels. Among other facts, the numerical results reveal that the resulting rates using both power allocation policies are quite similar, except for some specific cases that are discussed.

1. INTRODUCTION

In multiuser communications the channel resources can be allocated to different users in an infinite number of ways. Hence, the multiuser channel capacity is defined by a region which define all user rates that can be simultaneously supported by the channel with arbitrarily small error probability. Among all points of the capacity region, the point which achieves the sum capacity has special interest because it corresponds with the maximum total throughput. In broadcast channels the sum capacity has additional interest because it can be achieved by several practical multiple access strategies [1]: time division, frequency division or code division.

It is well known that the sum capacity of broadcast fading channels is achieved by transmitting only to the user with the best channel in each fading state. Therefore, the sum capacity of broadcast channels can be viewed as the capacity of an equivalent point-to-point SIMO (single-input multiple-output) channel with selection combining (SC) at the receiver, where the channel responses at the receiver branches are independent but differently distributed (i.d.d.) [2]. Moreover, the sum capacity is achieved by using optimal transmit power allocation as a function of the channel state. This optimal strategy is the so-called waterfilling power adaptation [3]. Although suboptimal in terms of capacity, the constant transmit power strategy is also of interest because it is simpler and, in many cases, its performance is close to the optimal power allocation.

Closed-form expressions for the ergodic capacity of point-to-point SIMO-SC Rayleigh channels were derived in [4] considering different power adaptation policies, but they assume identically distributed fading at the receiver branches. Ergodic capacity expressions for dependent and differently distributed SIMO-SC channels, assuming constant transmit power, were derived in [5] and [6], but they are restricted to dual-branch receivers. To the best of the authors' knowledge, there are not closed-form expressions for the ergodic capacity of SIMO-SC i.d.d. channels in the technical literature.

In this work, considering the equivalent point-to-point channel, we derive exact closed-form expressions for the sum capacity of downlink ergodic Rayleigh channels, assuming full channel side information. We consider two power allocation policies: optimal power adaptation and constant transmit power.

In asymmetric channels the strategy to achieve the sum capacity leads to different individual users' rates. We also derive exact closed-form expressions for the resulting individual users' rates assuming Rayleigh fading. Again we consider the cases of optimal power allocation and constant transmit power. These expressions give the coordinates of the sum capacity point on the ergodic capacity region.

The remainder of this paper is organized as follows. Section 2 describes the broadcast channel model. In section 3 we derive closed-form expressions for the sum capacity in the cases of optimal power allocation and constant power allocation. In section 4 we derive closed-form expressions for the individual users' rates under both power allocation policies. Numerical results are presented in section 5, followed by some conclusions.

2. BROADCAST CHANNEL MODEL

A narrowband broadcast channel with K users is considered. We assume that the transmitter and receivers have a single antenna. The transmitter is subject to an average power con-

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straint denoted by P. We assume independent and identically distributed (i.i.d.) AWGN noise at the Rx antennas, with single-sided power spectral density denoted by N_0 for all users. The receivers' bandwidth is denoted by B, so the noise power at the receivers is BN_0. The baseband-equivalent channel response between the transmitter and the k-th user is denoted by $h_k,\ k = 1\ldots,K$. We assume that the $|h_k|$ are i.d.d. Rayleigh distributed.

Let us denote $\gamma_k = |h_k|^2$. Since the $|h_k|$ are Rayleigh distributed, the γ_k will be exponentially distributed with cumulative distribution functions (c.d.f.) given by

$$F_k(x) = 1 - \exp\left(-x\, b_k\right)\ , \tag{1}$$

where b_k denotes the inverse of the average power gain for the k-th user: $\bar{\gamma}_k = \mathrm{E}\left\{\gamma_k\right\} = 1/b_k$. The probability density functions (p.d.f.) will be

$$f_k(x) = b_k\, \exp\left(-x\, b_k\right). \tag{2}$$

We assume, without loss of generality, that the channel is normalized so

$$\sum_{k=1}^{K} \bar{\gamma}_k = \sum_{k=1}^{K} \frac{1}{b_k} = K. \tag{3}$$

Under the above normalization, the average SNR at the users will be $\rho = P/BN_0$.

3. SUM CAPACITY

As it was mentioned the strategy to achieve the sum capacity is to transmit only to the user with the best channel for each channel state. Then, the sum capacity equals the capacity of an equivalent point-to-point SISO (single-input single-output) channel with power gain given by $\gamma = \max_k \left\{\gamma_k\right\}$ [2]. Since the γ_k are i.d.d., the c.d.f. of γ will be

$$F(x) = \prod_{k=1}^{K} F_k(x) = \prod_{k=1}^{K} \left[1 - \exp(-x\, b_k)\right]. \tag{4}$$

Expression (4) is a product of sums, which can be expressed as a sum of products as follows

$$F(x) = \sum_{\mathbf{i}\in S} (-1)^{\mathbf{i}\cdot\mathbf{1}} \exp\left(-x\, \mathbf{b}\cdot\mathbf{i}\right), \tag{5}$$

where $\mathbf{b} = [b_1\, b_2 \ldots b_K]^T$, $\mathbf{1}$ denotes the all-ones K-dimensional vector and S is the set of all K-dimensional vectors with entries taking values 0 or 1. In other words, S contains the 2^K binary words of length K. From (5), the p.d.f. of γ will be

$$f(x) = -\sum_{\mathbf{i}\in S} (-1)^{\mathbf{i}\cdot\mathbf{1}} \left(\mathbf{b}\cdot\mathbf{i}\right)\, \exp\left(-x\, \mathbf{b}\cdot\mathbf{i}\right). \tag{6}$$

3.1. Constant Transmit Power

Assuming that the channel is ergodic, the capacity of the equivalent point-to-point channel is given by [3]

$$\begin{aligned} C &= \mathrm{E}\left[\log_2\left(1 + \gamma\, \rho\right)\right] = \\ &= \int_0^\infty \log_2\left(1 + x\, \rho\right) f(x)\, dx. \end{aligned} \tag{7}$$

Considering (6), the sum capacity of the broadcast channel can be expressed as follows

$$C_{sum} = -\sum_{\mathbf{i}\in S,\ \mathbf{i}\neq\mathbf{0}} (-1)^{\mathbf{i}\cdot\mathbf{1}} \frac{\exp\left(\mathbf{b}\cdot\mathbf{i}/\rho\right)}{\ln 2}\, E_1\left(\mathbf{b}\cdot\mathbf{i}/\rho\right), \tag{8}$$

where $E_1(\cdot)$ is the exponential integral. When the entries of $\mathbf{b}$ are identical (i.i.d. channel), (8) coincides with the expression derived in [4] for the ergodic capacity of a single-user i.i.d. SIMO-SC Rayleigh channel.

3.2. Optimal Power Allocation

Assuming optimal power allocation the capacity of the equivalent point-to-point fading channel is given by [3]

$$C = \max_{p(x)} \int_0^\infty \log_2\left(1 + x\, \rho\, p(x)\right)\ f(x)\, dx, \tag{9}$$

subject to

$$\int_0^\infty p(x)\, f(x)\, dx = 1, \tag{10}$$

where $p(x)$ denotes the optimal power allocation, normalized to the average power P, as a function of the channel power gain. The solution to this well-known optimization problem is the so-called waterfilling [7], which is given by

$$p(x) = \left(\frac{1}{\lambda} - \frac{1}{\rho\, x}\right)^+, \tag{11}$$

where $y^+ = \max(y, 0)$ and the parameter λ depends on the distribution of γ as follows

$$\int_{\lambda/\rho}^\infty \left(\frac{1}{\lambda} - \frac{1}{\rho\, x}\right) f(x)\, dx = 1. \tag{12}$$

Note that, according to (11), when $\gamma < \lambda/\rho$ the transmission is suspended. Then, there is a probability of outage given by $F(\lambda/\rho)$. Substituting (11) in (9), the capacity reduces to

$$C = \int_{\lambda/\rho}^\infty \log_2\left(\frac{\rho\, x}{\lambda}\right)\ f(x)\, dx. \tag{13}$$

Substituting (6) in (13) the sum capacity of the broadcast channel can be expressed as follows

$$C_{sum} = -\sum_{\mathbf{i}\in S,\mathbf{i}\neq\mathbf{0}} (-1)^{\mathbf{i}\cdot\mathbf{1}} \frac{1}{\ln 2}\, E_1(\mathbf{b}\cdot\mathbf{i}\ \ \lambda/\rho). \tag{14}$$

Considering (6) and (5), the integral equation (12) reduces to the following implicit equation

$$\begin{aligned}\lambda &= \sum_{\mathbf{i}\in S,\mathbf{i}\neq\mathbf{0}} (-1)^{\mathbf{i}\cdot\mathbf{1}} \\ &\times \left[\frac{\lambda\,\mathbf{b}\cdot\mathbf{i}}{\rho} E_1\left(\frac{\lambda\,\mathbf{b}\cdot\mathbf{i}}{\rho}\right) - \exp\left(\frac{-\lambda\,\mathbf{b}\cdot\mathbf{i}}{\rho}\right)\right].\end{aligned} \tag{15}$$

And the probability of outage

$$P_{out} = F(\lambda/\rho) = \sum_{\mathbf{i}\in S}(-1)^{\mathbf{i}\cdot\mathbf{1}} \exp\left(-\mathbf{b}\cdot\mathbf{i}\ \lambda/\rho\right). \tag{16}$$

When the entries of $\mathbf{b}$ are identical (i.i.d. channel), the expressions (14), (15) and (16) reduce to the expressions derived in [4] for the case of single-user i.i.d. SIMO-SC Rayleigh channels.

4. INDIVIDUAL USERS' RATES

Since the channel is asymmetric, the achievable user rates will be different. To derive the expression for the users' rates, let define the following effective channel gain for the s-th user

$$\gamma_s^* = \begin{cases} 0\,, & \gamma_s < \gamma_{-s} \\ \gamma_s\,, & \gamma_s > \gamma_{-s} \end{cases} \tag{17}$$

where $\gamma_{-s} = \max_{k\neq s}\{\gamma_k\}$. The p.d.f. of γ_s^* can be expressed as follows

$$f_s^*(x) = \text{Prob}\{\gamma_s < \gamma_{-s}\}\ \delta(x) + f_s(x)\,F_{-s}(x), \tag{18}$$

where $\delta(x)$ is the Dirac delta function, $f_s(x)$ is given by (2) and $F_{-s}(x)$ is the c.d.f. of γ_{-s}, which is given by

$$\begin{aligned}F_{-s}(x) &= \prod_{k\neq s}^{K}\left[1-\exp(-x\,b_k)\right] \\ &= \sum_{\mathbf{i}\in S}(-1)^{\mathbf{i}\cdot\mathbf{1}}\,(1-i_s)\,\exp\left(-x\,\mathbf{b}\cdot\mathbf{i}\right),\end{aligned} \tag{19}$$

where i_s denotes the s-th component of $\mathbf{i}$. Considering (19) and (2),

$$f_s(x)\,F_{-s}(x) = -\sum_{\mathbf{i}\in S}(-1)^{\mathbf{i}\cdot\mathbf{1}}\,(b_s\,i_s)\,\exp\left(-x\,\mathbf{b}\cdot\mathbf{i}\right). \tag{20}$$

4.1. Constant Transmit Power

The rate for the s-th user will be the capacity of the effective point-to-point channel with power gain γ_s^*. Then,

$$\begin{aligned}R_s &= \int_0^\infty \log_2(1+\rho\,\gamma)\,f_s^*(x)\,dx \\ &= \int_0^\infty \log_2(1+\rho\,x)\,f_s(x)\,F_{-s}(x)\,dx.\end{aligned} \tag{21}$$

Substituting (20) in (21), the rate for the s-th user can be expressed as follows

$$R_s = -\sum_{\mathbf{i}\in S,\,\mathbf{i}\neq\mathbf{0}}(-1)^{\mathbf{i}\cdot\mathbf{1}}\frac{(b_s\,i_s)\,\exp(\mathbf{b}\cdot\mathbf{i}/\rho)}{(\mathbf{b}\cdot\mathbf{i})\,\ln 2}\,E_1(\mathbf{b}\cdot\mathbf{i}/\rho). \tag{22}$$

Note that the sum of the users' rates coincides with the sum capacity given by (8).

4.2. Optimal Power Allocation

Considering the effective channel gain γ_s^*, the rate of the s-th user will be

$$R_s = \int_0^\infty \log_2\left(1+x\,\rho\,p(x)\right)\,f_s^*(x)\,dx. \tag{23}$$

Substituting (11) and (18) in (23)

$$R_s = \int_{\lambda/\rho}^\infty \log_2\left(\frac{\rho\,x}{\lambda}\right)\,f_s(x)\,F_{-s}(x)\,dx, \tag{24}$$

where λ is given by (15). Finally, substituting (20) in (24)

$$R_s = -\sum_{\mathbf{i}\in S,\,\mathbf{i}\neq\mathbf{0}}(-1)^{\mathbf{i}\cdot\mathbf{1}}\frac{(b_s\,i_s)}{(\mathbf{b}\cdot\mathbf{i})\,\ln 2}\,E_1(\mathbf{b}\cdot\mathbf{i}\,\lambda/\rho). \tag{25}$$

As it is expected, the sum of the users' rates coincides with the sum capacity given by (14).

5. NUMERICAL RESULTS

Expressions (22) and (25) give the achievable users' rates for a given broadcast channel distribution defined by $\mathbf{b}$. In other words, they define the vector transformation $\mathbf{b} \longrightarrow \mathbf{R}$, where the entries of $\mathbf{R}$ are the individual users' rates. Hereafter, we present the rate vectors $\mathbf{R}$ obtained for different specific vectors $\mathbf{b}$.

As first example, we consider a broadcast channel where the average channel gains are linearly distributed according to:

$$\bar{\gamma}_k = a\,k \Rightarrow b_k = k^{-1}/a\,,\ k = 1,\ldots,\,K, \tag{26}$$

where a is a constant determined by the channel normalization of (3): $a = 2/(K+1)$.

Figure 1 shows the sum capacity, as a function of the average SNR (ρ), for different number of users (K) and the two power allocation policies: constant transmit power and optimal power allocation. As in point-to-point communications, the gain of waterfilling over the constant power policy is very small, except at the low SNR regime. But, even for low SNR, this gain is negligible except for very low number of users. Since the base-station transmits always to the best user the gain variations of the equivalent point-to-point channel are very low when K is high. Therefore, the optimal power allocation has no room for exploiting the channel variation along the time. The figure also shows that the sum capacity improves monotonically with the number of users (K), but, as K increases, the improvement is lower.

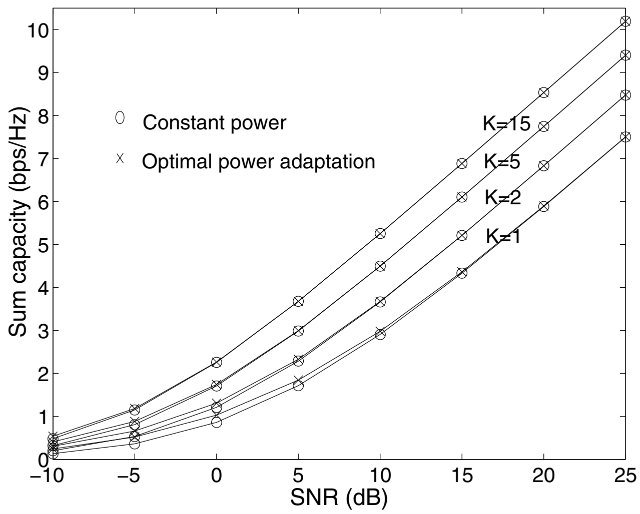

Fig. 1. Sum capacity, as a function of the average SNR (ρ), for different number of users.

Figure 2 shows the resulting individual users' rates, as a function of the average SNR (ρ), when the number of users is $K = 10$, assuming the average channels' gains defined by (26). Curves for constant transmit power and optimal power adaptation are shown. Even at the low SNR regime, the gain of the optimal power adaptation is very low for all users. The rates for the worse users remain very low at the high SNR-regime, whereas, for the better channels, they increases significantly with the average SNR. In other words, the better the channel, the higher is the rate improvement with the average SNR.

Figure 3 refers to a downlink scenario with $K = 10$ users, where the users' channels are grouped in two sets. In each set the average power gain of the channels are identical. Therefore,

$$\bar{\gamma}_k = \begin{cases} a\,,\ k = 1,\ldots,5 \\ a\,\Delta\,,\ k = 6,\ldots,10 \end{cases}$$

$$\Rightarrow b_k = \begin{cases} 1/a\,,\ k = 1,\ldots,5 \\ 1/(a\,\Delta)\,,\ k = 6,\ldots,10 \end{cases} \tag{27}$$

Considering the channel normalization (3): $a = 2/(\Delta + 1)$. All the users in a set will have the same rate. Figure 3 shows the rates for the users in each set, as a function of the parameter Δ, for average SNR $\rho = 0$ and $\rho = 10$ dB. Note that Δ determines the difference of the average gain between the two sets of users. As it is expected, when Δ grows, the individual rates for the users of the second set increase, whereas the rates for the users of the first set decrease. In this case, constant transmitted power is assumed. The results for optimal power allocation are nearly identical.

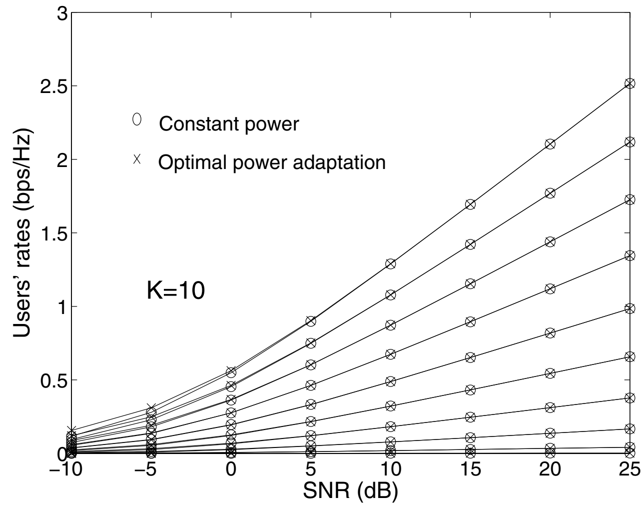

Fig. 2. Individual users' rates as a function of the average SNR (ρ).

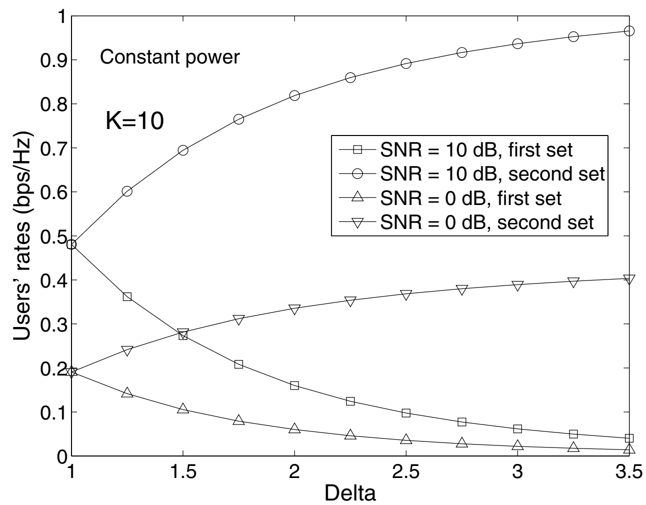

Fig. 3. Individual users' rates as a function of parameter Δ.

Now, we consider the case of $K = 10$ users, where one average channel gain is dominant whereas the rest of the channels have the same average power gain:

$$\bar{\gamma}_k = \begin{cases} a\,\Delta\,,\ k = 1 \\ a\,,\ k = 2,\ldots,10 \end{cases}$$

$$\Rightarrow b_k = \begin{cases} 1/(a\,\Delta)\,,\ k = 1 \\ 1/a\,,\ k = 2,\ldots,10 \end{cases} \tag{28}$$

Considering the channel normalization (3): $a = K/(\Delta + K - 1)$. Note that the parameter Δ determines the difference of the average gain between the dominant channel and the rest of channels. Figure 4 shows the rates for the two sets of users, as a function of Δ, for $\rho = 10$ dB and $\rho = 0$

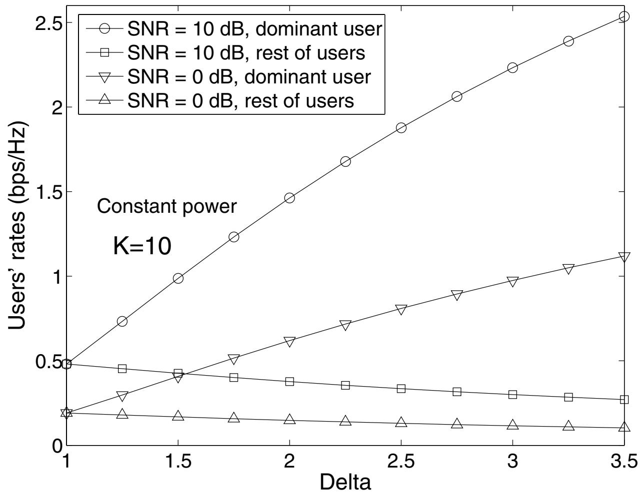

Fig. 4. Individual users' rates as a function of parameter Δ.

dB. As it is expected, the rate of the dominant user increases significantly with Δ, whereas the rates of the rest of users decrease slowly. Only the rates for constant power transmission are shown. The rates for optimal power allocation are nearly identical for these values of ρ.

In the low SNR regime, when one user is dominant, the gain of the optimal power adaptation can be important. This fact is shown in figure 5. In this case the average channel gains are as in (28), but the average SNR is $\rho = -5$ dB. As it is expected, for the dominant user, the gap in rate between both power allocation policies is remarkable and increases with Δ. On the other hand, for the rest of users, both power allocation policies produce similar rates.

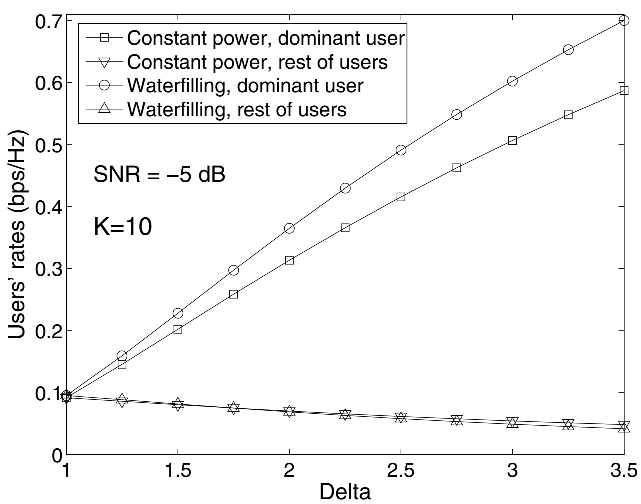

Fig. 5. Individual users' rates as a function of parameter Δ.

6. CONCLUSIONS

We have derived closed-form expressions for the sum capacity of broadcast ergodic Rayleigh fading channels considering two power allocation policies along time: constant transmit power and optimal power allocation. In addition, we have derived closed-form expressions for the individual users' rates when the sum capacity is achieved, for both power allocation policies. In the case of optimal power allocation we have also derived a closed-form expression for the outage probability. These expressions have been used to obtain the sum capacity and individual rates in a variety of broadcast channels. Among other facts, the numerical results reveal that the resulting rates using both power allocation policies are quite similar, except for the rate of a dominant user channel at the low SNR regime.

7. REFERENCES

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