Independent Component Analysis of Quaternion Gaussian Vectors

Javier Via, Luis Vielva, Ignacio Santamaria
Dept. of Communications Engineering
University of Cantabria. Spain
Email: {jvia,luis,nacho}@gtas.dicom.unican.es

Daniel P. Palomar
Dept. of Electronic and Computer Engineering
Hong Kong University of Science and Technology
Email: palomar@ust.hk

***Abstract*—This paper addresses the independent component analysis (ICA) of quaternion Gaussian vectors. Firstly, we define the *properness profile* of a quaternion random variable, which can be seen as the quaternion analogue of the circularity coefficients of complex vectors. The properness profile is a three-dimensional pure quaternion vector, which does not only measure the improperness degree of the quaternion random variable, but also provides the improperness distribution. Secondly, we prove that the quaternion ICA model can be identified up to the trivial scale and permutation ambiguities, and a residual quaternion mixture among the sources with *rotationally equivalent properness profiles*, i.e., properness profiles related by a quaternion rotation. Finally, the main results of the paper are illustrated by means of some numerical examples.**

I. Introduction

In the last years, quaternion signal processing has received increasing attention due, among others, to its successful application in image processing [1], wind modeling [2], and design of space-time block codes [3], [4]. This has motivated the extension of several signal processing techniques to the case of quaternionic signals, as well as a rigorous second-order statistical analysis of quaternion random vectors [5], [6].

In this paper we consider the independent component analysis (ICA) [7] of quaternion random vectors, which can find almost-direct applications in problems such as blind decoding in Alamouti-based multiuser systems, blind separation of color images, or the intrinsical analysis of wind profiles. In particular, we focus on the fundamental case of quaternion Gaussian vectors, and derive the necessary and sufficient conditions for the identifiability of the quaternion ICA model. Analogously to the complex Gaussian case [8]–[10], quaternion ICA relies on the independence and improperness of the sources. Thus, the identifiability analysis is based on a new statistical measure, the properness profile, which can can be seen as the quaternion counterpart of the circularity coefficients of complex vectors [10]. However, the properness profile of a quaternion random variable, which is a three-dimensional pure quaternion vector, does not only measure the improperness degree, but also indicates the improperness distribution. As a consequence, the quaternion ICA model can be unambiguously identified up to the trivial indeterminacies, consisting in arbitrary quaternion scale factors and permutations, and a residual quaternion mixture among the sources with rotationally equivalent properness profiles, i.e., with properness profiles related by a three-dimensional rotation. Additionally, the theoretical results are illustrated by means of some simulation examples, which also pose some interesting questions for future research.

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II. Preliminaries

Throughout this paper we will use bold-faced upper case letters to denote matrices, bold-faced lower case letters for column vectors, and light-faced lower case letters for scalar quantities. Superscripts $(\cdot)^T$ and $(\cdot)^H$ denote transpose and Hermitian (i.e., transpose and quaternion conjugate), respectively. The notation $\mathbf{A} \in \mathbb{R}^{m\times n}$ (respectively $\mathbf{A} \in \mathbb{H}^{m\times n}$) means that $\mathbf{A}$ is a real (respectively quaternion) $m \times n$ matrix. $\mathbf{I}_n$ is the identity matrix of dimension n, $\mathbf{0}_{m\times n}$ is the $m \times n$ zero matrix, and diag($\mathbf{a}$) denotes the diagonal matrix with vector $\mathbf{a}$ along its diagonal. Finally, E is the expectation operator, and in general $\mathbf{R}_{\mathbf{a},\mathbf{b}}$ is the cross-correlation matrix for vectors $\mathbf{a}$ and $\mathbf{b}$, i.e., $\mathbf{R}_{\mathbf{a},\mathbf{b}} = E\mathbf{a}\mathbf{b}^H$.

A. Quaternion Algebra

Quaternions are hypercomplex numbers defined by

$$x = r_1 + \eta r_\eta + \eta' r_{\eta'} + \eta'' r_{\eta''},$$

where $r_1, r_\eta, r_{\eta'}, r_{\eta''} \in \mathbb{R}$ are four real numbers, and the three imaginary units[1] (η, η', η'') satisfy

$$\eta^2 = \eta'^2 = \eta''^2 = \eta\eta'\eta'' = -1,$$

which also implies

$$\eta\eta' = \eta'', \qquad \eta'\eta'' = \eta, \qquad \eta''\eta = \eta'.$$

Quaternions form a skew field $\mathbb{H}$ [11], which means that they satisfy the axioms of a field except the commutative law of the product, i.e., for $x, y \in \mathbb{H}$, $xy \neq yx$ in general. The conjugate of a quaternion x is $x^* = r_1 - \eta r_\eta - \eta' r_{\eta'} - \eta'' r_{\eta''}$, and the inner product of two quaternions x, y is defined as the real part of xy^*. Two quaternions are orthogonal if and only if (iff) their inner product is zero, and the norm of a quaternion x is $|x| = \sqrt{xx^*} = \sqrt{r_1^2 + r_\eta^2 + r_{\eta'}^2 + r_{\eta''}^2}$. Furthermore, we say that ν is a pure unit quaternion iff $\nu^2 = -1$ (i.e., iff $|\nu| = 1$ and its real part is zero). Quaternions also admit the Euler representation

$$x = |x|e^{\nu\theta} = |x|\left(\cos\theta + \nu\sin\theta\right),$$

where ν is a pure unit quaternion and $\theta \in \mathbb{R}$ is the angle (or argument) of the quaternion.

Definition 1 (Quaternion Rotation and Involution [11]): Consider a non-zero quaternion $a = |a|e^{\nu\theta} = |a|\left(\cos\theta + \nu\sin\theta\right)$, then[2]

$$x^{(a)} = axa^{-1},$$

[1] A particular choice of the imaginary axes is the canonical basis $\{i, j, k\}$. However, in this paper we use the more general representation $\{\eta, \eta', \eta''\}$.

[2] From now on, we will use the notation $\mathbf{A}^{(a)}$ to denote the element-wise rotation of matrix $\mathbf{A}$.

represents a three-dimensional rotation of the imaginary part of x. Specifically, the vector $[r_\eta, r_{\eta'}, r_{\eta''}]^T$ is rotated clockwise an angle 2θ in the pure imaginary plane orthogonal to ν. In the particular case of pure quaternions ν, $x^{(\nu)}$ represents a rotation of angle π, which is an involution.

B. Second Order Statistics of Quaternion Random Vectors

Analogously to the case of complex vectors, the statistical analysis of a quaternion random vector $\mathbf{x} \in \mathbb{H}^{n\times 1}$ can be directly based on its real representation $\mathbf{r_x} = \left[\mathbf{r}_1^T, \mathbf{r}_\eta^T, \mathbf{r}_{\eta'}^T, \mathbf{r}_{\eta''}^T\right]^T$. However, we can get more insight on its statistical properties by introducing the augmented quaternion vector $\bar{\mathbf{x}} = \left[\mathbf{x}^T, \mathbf{x}^{(\eta)^T}, \mathbf{x}^{(\eta')^T}, \mathbf{x}^{(\eta'')^T}\right]^T$. Thus, the second-order statistics (SOS) of a quaternion random vector are given by the augmented covariance matrix

$$\mathbf{R}_{\bar{\mathbf{x}},\bar{\mathbf{x}}} = E\bar{\mathbf{x}}\bar{\mathbf{x}}^H = \begin{bmatrix} \mathbf{R}_{\mathbf{x},\mathbf{x}} & \mathbf{R}_{\mathbf{x},\mathbf{x}^{(\eta)}} & \mathbf{R}_{\mathbf{x},\mathbf{x}^{(\eta')}} & \mathbf{R}_{\mathbf{x},\mathbf{x}^{(\eta'')}} \\ \mathbf{R}^{(\eta)}_{\mathbf{x},\mathbf{x}^{(\eta)}} & \mathbf{R}^{(\eta)}_{\mathbf{x},\mathbf{x}} & \mathbf{R}^{(\eta)}_{\mathbf{x},\mathbf{x}^{(\eta'')}} & \mathbf{R}^{(\eta)}_{\mathbf{x},\mathbf{x}^{(\eta')}} \\ \mathbf{R}^{(\eta')}_{\mathbf{x},\mathbf{x}^{(\eta')}} & \mathbf{R}^{(\eta')}_{\mathbf{x},\mathbf{x}^{(\eta'')}} & \mathbf{R}^{(\eta')}_{\mathbf{x},\mathbf{x}} & \mathbf{R}^{(\eta')}_{\mathbf{x},\mathbf{x}^{(\eta)}} \\ \mathbf{R}^{(\eta'')}_{\mathbf{x},\mathbf{x}^{(\eta'')}} & \mathbf{R}^{(\eta'')}_{\mathbf{x},\mathbf{x}^{(\eta')}} & \mathbf{R}^{(\eta'')}_{\mathbf{x},\mathbf{x}^{(\eta)}} & \mathbf{R}^{(\eta'')}_{\mathbf{x},\mathbf{x}} \end{bmatrix},$$

where we can readily identify the covariance matrix $\mathbf{R}_{\mathbf{x},\mathbf{x}} = E\mathbf{x}\mathbf{x}^H$ and three complementary covariance matrices $\mathbf{R}_{\mathbf{x},\mathbf{x}^{(\eta)}} = E\mathbf{x}\mathbf{x}^{(\eta)^H}$, $\mathbf{R}_{\mathbf{x},\mathbf{x}^{(\eta')}} = E\mathbf{x}\mathbf{x}^{(\eta')^H}$ and $\mathbf{R}_{\mathbf{x},\mathbf{x}^{(\eta'')}} = E\mathbf{x}\mathbf{x}^{(\eta'')^H}$. From these four matrices, we can define three different kinds of quaternion properness [5]. In particular, in this paper we consider the strongest kind of quaternion properness, which is defined as follows.[3]

Definition 2 (Quaternion $\mathbb{Q}$-Properness [5], [6]): A quaternion random vector $\mathbf{x}$ is $\mathbb{Q}$-proper iff the three complementary covariance matrices $\mathbf{R}_{\mathbf{x},\mathbf{x}^{(\eta)}}$, $\mathbf{R}_{\mathbf{x},\mathbf{x}^{(\eta')}}$ and $\mathbf{R}_{\mathbf{x},\mathbf{x}^{(\eta'')}}$ vanish.

The main practical implication of $\mathbb{Q}$-properness consists in the fact that, for a $\mathbb{Q}$-proper random vector $\mathbf{x} \in \mathbb{H}^{n\times 1}$, the optimal linear processing is quaternion linear, i.e., it takes the form $\mathbf{u} = \mathbf{F}_1^H\mathbf{x}$ instead of the more general *widely-linear* transformation $\mathbf{u} = \mathbf{F}_1^H\mathbf{x} + \mathbf{F}_\eta^H\mathbf{x}^{(\eta)} + \mathbf{F}_{\eta'}^H\mathbf{x}^{(\eta')} + \mathbf{F}_{\eta''}^H\mathbf{x}^{(\eta'')}$ [5], [6].

III. Quaternion Independent Component Analysis

In this section we introduce the independent component analysis (ICA) model for quaternion vectors, and derive the necessary and sufficient conditions for the identifiability of the ICA model.

A. ICA Model

Consider a quaternion random vector $\mathbf{s} \in \mathbb{H}^{m\times 1}$ representing m independent source signals, which are mixed by a full-column rank mixing matrix $\mathbf{A} \in \mathbb{H}^{n\times m}$ ($n \geq m$). That is, we have the model $\mathbf{x} = \mathbf{As}$, where $\mathbf{x} \in \mathbb{H}^{n\times 1}$ is a quaternion random vector representing the available observations.

Analogously to the case of real or complex vectors [7], [10], the quaternion ICA model is affected by two trivial indeterminacies, which consist in a quaternion scale factor and a permutation of the sources $\mathbf{s}$ and columns of the mixing matrix $\mathbf{A}$. Therefore, we can assume without loss of generality that the sources are unit-variance quaternion random variables with diagonal complementary covariance matrices $\mathbf{\Lambda}_\eta = E\mathbf{s}\mathbf{s}^{(\eta)^H}$, $\mathbf{\Lambda}_{\eta'} = E\mathbf{s}\mathbf{s}^{(\eta')^H}$, $\mathbf{\Lambda}_{\eta''} = E\mathbf{s}\mathbf{s}^{(\eta'')^H}$.

With the above assumptions and considering the case of Gaussian data (or equivalently, limiting our analysis to SOS-based techniques), the ICA problem amounts to finding the mixing matrix $\mathbf{A}$ and the complementary covariance matrices of the sources satisfying

$$\mathbf{A}\mathbf{A}^H = \mathbf{R}_{\mathbf{x},\mathbf{x}}, \qquad \mathbf{A}\mathbf{\Lambda}_\nu\mathbf{A}^{(\nu)^H} = \mathbf{R}_{\mathbf{x},\mathbf{x}^{(\nu)}}, \qquad (1)$$

[3] See [12], [13] for closely related, but different, $\mathbb{Q}$-properness definitions.

for all pure unit quaternions ν. Equivalently, taking into account that the complementary covariance matrix $\mathbf{R}_{\mathbf{x},\mathbf{x}^{(\nu)}}$ (for all pure unit quaternions ν) can be written as a quaternion linear combination of $\mathbf{R}_{\mathbf{x},\mathbf{x}^{(\eta)}}$, $\mathbf{R}_{\mathbf{x},\mathbf{x}^{(\eta')}}$ and $\mathbf{R}_{\mathbf{x},\mathbf{x}^{(\eta'')}}$ (for an orthogonal basis $\{\eta, \eta', \eta''\}$) [5], [6], eq. (1) can be rewritten as

$$\mathbf{A}\mathbf{A}^H = \mathbf{R}_{\mathbf{x},\mathbf{x}}, \qquad \mathbf{A}\mathbf{\Lambda}_\eta\mathbf{A}^{(\eta)^H} = \mathbf{R}_{\mathbf{x},\mathbf{x}^{(\eta)}},$$
$$\mathbf{A}\mathbf{\Lambda}_{\eta'}\mathbf{A}^{(\eta')^H} = \mathbf{R}_{\mathbf{x},\mathbf{x}^{(\eta')}}, \qquad \mathbf{A}\mathbf{\Lambda}_{\eta''}\mathbf{A}^{(\eta'')^H} = \mathbf{R}_{\mathbf{x},\mathbf{x}^{(\eta'')}}.$$

B. Identifiability Conditions

Analogously to the case of complex vectors [10], the identifiability of the quaternion ICA model from SOS relies on the improperness of the sources. Here, we start by introducing the following definitions.

Definition 3 (Properness Profile): The properness profile of a quaternion random variable s is defined as

$$\boldsymbol{\psi}_s = \begin{bmatrix} \psi_{s,\eta} \\ \psi_{s,\eta'} \\ \psi_{s,\eta''} \end{bmatrix} = \begin{bmatrix} \lambda_{s,\eta}\eta \\ \lambda_{s,\eta'}\eta' \\ \lambda_{s,\eta''}\eta'' \end{bmatrix},$$

where, for a pure unit quaternion ν, $\lambda_{s,\nu} = Ess^{*(\nu)}/E|s|^2$ is the (normalized) complementary variance.

Definition 4 (Rotationally Equivalent Properness Profiles): The properness profiles of two quaternion random variables s_1, s_2 are rotationally equivalent iff they are related by a quaternion rotation, i.e., iff there exists a quaternion a such that

$$\boldsymbol{\psi}_{s_2} = \boldsymbol{\psi}_{s_1}^{(a)}.$$

Interestingly, the properness profile is a pure quaternion vector, and therefore each of its entries can be seen as one point in a three-dimensional space. Furthermore, it is easy to prove that the rotational equivalence between two properness profiles does not depend on the particular quaternion basis, i.e., if $\boldsymbol{\psi}_{s_1}$ and $\boldsymbol{\psi}_{s_2}$ are rotationally equivalent in a basis $\{\eta, \eta', \eta''\}$, then they are rotationally equivalent for all the orthogonal bases $\{\nu, \nu', \nu''\}$. On the other hand, it is also important to note that a quaternion scale factor $s_2 = as_1$, which is one of the trivial ambiguities in the quaternion ICA model, results in a rotation of the properness profile $\boldsymbol{\psi}_{s_2} = \boldsymbol{\psi}_{s_1}^{(a)}$. Finally, we can introduce the following theorem, which states the quaternion ICA identifiability conditions.

Theorem 1 (ICA Identifiability): Given the ICA model $\mathbf{x} = \mathbf{As}$, with independent entries in $\mathbf{s}$ and full-column rank $\mathbf{A}$, the sources $\mathbf{s}$ and the mixing matrix $\mathbf{A}$ can be recovered from the SOS of the observations up to the following ambiguities:

- A permutation and quaternion scale factor.
- A residual quaternion linear mixture affecting the sources with rotationally equivalent properness profiles.

Proof: Let us start by noting that, as a consequence of (1), all the solutions $\hat{\mathbf{A}} \in \mathbb{H}^{n\times m}$ of the quaternion ICA model can be written as $\hat{\mathbf{A}} = \mathbf{AQ}$, where $\mathbf{A}$ is the actual mixing matrix and $\mathbf{Q} \in \mathbb{H}^{m\times m}$ is a unitary quaternion matrix. Furthermore, using the trivial ambiguities, we can introduce a permutation and a quaternion scale factor in the columns of $\hat{\mathbf{A}}$ to ensure that the elements in the diagonal of $\mathbf{Q}$ are positive real numbers. Thus, from (1) we can see that a mixing matrix $\hat{\mathbf{A}} = \mathbf{AQ}$, and a set of complementary covariance matrices $\hat{\mathbf{\Lambda}}_\nu$, are solutions of the quaternion ICA model iff, for all ν, $\mathbf{Q}^H\mathbf{\Lambda}_\nu\mathbf{Q}^{(\nu)} = \hat{\mathbf{\Lambda}}_\nu$, or equivalently

$$\mathbf{Q}^H\mathbf{\Psi}_\nu\mathbf{Q} = \hat{\mathbf{\Psi}}_\nu, \qquad \forall\nu, \qquad (2)$$

where $\mathbf{\Psi}_\nu = \mathbf{\Lambda}_\nu\nu = \mathrm{diag}([\psi_{s_1,\nu}, \ldots, \psi_{s_m,\nu}])$ contains one of the elements of the properness profiles of all the sources, and $\hat{\mathbf{\Psi}}_\nu$ is defined analogously.

Let us now focus on the first row and column of the ambiguity matrix $\mathbf{Q}$. In particular, we will write

$$\mathbf{Q} = \begin{bmatrix} q_1 & \mathbf{v}^H \\ \mathbf{w} & \mathbf{Q}_{-1} \end{bmatrix},$$

where q_1 is a real and positive number, $\mathbf{v}, \mathbf{w} \in \mathbb{H}^{(m-1)\times 1}$, and $\mathbf{Q}_{-1} \in \mathbb{H}^{(m-1)\times(m-1)}$. Here, it is clear that the unitarity of $\mathbf{Q}$ implies

$$\mathbf{v}q_1 + \mathbf{Q}_{-1}^H \mathbf{w} = \mathbf{0}_{(m-1)\times 1}, \tag{3}$$

$$q_1^2 + \|\mathbf{w}\|^2 = 1, \tag{4}$$

$$\mathbf{w}\mathbf{w}^H + \mathbf{Q}_{-1}\mathbf{Q}_{-1}^H = \mathbf{I}_{m-1}. \tag{5}$$

Analogously, the diagonal matrix $\mathbf{\Psi}_\nu$ can be written as $\mathbf{\Psi}_\nu = \text{diag}([\psi_{s_1,\nu}, \boldsymbol{\psi}_{s_{-1},\nu}^T]^T)$, where $\boldsymbol{\psi}_{s_{-1},\nu} = [\psi_{s_2,\nu}, \ldots, \psi_{s_m,\nu}]$. Thus, considering the first column of $\hat{\mathbf{\Psi}}_\nu$, we can see that (2) yields

$$\mathbf{v}\psi_{s_1,\nu}q_1 + \mathbf{Q}_{-1}^H \text{diag}(\boldsymbol{\psi}_{s_{-1},\nu})\mathbf{w} = \mathbf{0}_{(m-1)\times 1}, \qquad \forall\nu.$$

Moreover, taking into account the property $ab = ba^{(b^*)}$, and noting that q_1 is a real scalar, the above equation can be rewritten as

$$\mathbf{v}q_1\psi_{s_1,\nu} + \mathbf{Q}_{-1}^H \text{diag}(\mathbf{w})\boldsymbol{\psi}_{s_{-1},\nu}^{(\mathbf{w}^*)} = \mathbf{0}_{(m-1)\times 1}, \qquad \forall\nu,$$

where, with a slight abuse of notation, $\boldsymbol{\psi}_{s_{-1},\nu}^{(\mathbf{w}^*)}$ denotes the element-wise rotation of the entries in $\boldsymbol{\psi}_{s_{-1},\nu}$. Now, defining $\mathbf{1} \in \mathbb{R}^{(m-1)\times 1}$ as the vector of ones, and using (3) we have

$$\mathbf{Q}_{-1}^H \text{diag}(\mathbf{w}) \left(\boldsymbol{\psi}_{s_{-1},\nu}^{(\mathbf{w}^*)} - \mathbf{1}\psi_{s_1,\nu} \right) = \mathbf{0}_{(m-1)\times 1}, \qquad \forall\nu.$$

Additionally, noting that $q_1 > 0$, the combination of (4) and (5) ensures that $\mathbf{Q}_{-1}$ is invertible, which yields

$$\text{diag}(\mathbf{w}) \left(\boldsymbol{\psi}_{s_{-1},\nu}^{(\mathbf{w}^*)} - \mathbf{1}\psi_{s_1,\nu} \right) = \mathbf{0}_{(m-1)\times 1}, \qquad \forall\nu,$$

or equivalently, for $k = 2, \ldots, m$

$$q_{k,1}(\psi_{s_k,\nu}^{(q_{k,1}^*)} - \psi_{s_1,\nu}) = 0, \qquad \forall\nu,$$

where $q_{k,1}$ is the k-th element in the first column of $\mathbf{Q}$. Therefore, since the above equation holds for all ν, we can conclude that if the properness profiles $\boldsymbol{\psi}_{s_1}$ and $\boldsymbol{\psi}_{s_k}$ are not rotationally equivalent, then $q_{k,1} = 0$.

Finally, following the same reasoning for the remaining rows and columns of $\mathbf{Q}$ we can see that, excluding the trivial ambiguities, the only possible indeterminacies are given by a unitary quaternion matrix affecting the sources with rotationally equivalent properness profiles. In fact, assuming a set of K sources $\mathbf{s} = [s_1, \ldots, s_K]^T$ with rotationally equivalent properness profiles

$$\boldsymbol{\psi}_{s_1} = \boldsymbol{\psi}_{s_2}^{(a_2)} = \cdots = \boldsymbol{\psi}_{s_K}^{(a_K)},$$

we can easily see that the associated matrix $\mathbf{\Psi}_\nu = \text{diag}([\psi_{s_1,\nu}, \ldots, \psi_{s_K,\nu}])$ can be written as

$$\mathbf{\Psi}_\nu = \text{diag}(\mathbf{a})\psi_{s_1,\nu}\text{diag}(\mathbf{a})^H, \qquad \forall\nu,$$

where $\mathbf{a} = [1, a_2/|a_2|, \ldots, a_K/|a_K|]$. Therefore, any linear transformation of the form

$$\hat{\mathbf{s}} = \tilde{\mathbf{Q}}\text{diag}(\mathbf{a})^H \mathbf{s},$$

with $\tilde{\mathbf{Q}} \in \mathbb{R}^{K\times K}$ a real unitary matrix, will satisfy the ambiguity condition in eq. (2), i.e., the indeterminacy affecting the sources with rotationally equivalent properness profiles can not be avoided without exploiting some additional property of the sources or the mixing matrix. ■

Theorem 1 shows that the properness profiles play a crucial role in the identifiability of the quaternion ICA model. That is, the properness profiles can be seen as the quaternion counterpart of the circularity coefficients of complex vectors [10]. However, we must point out two key differences with the complex case:

- In the complex case, the ICA identifiability conditions can be reformulated in terms of the improperness degree of the sources. That is, we can say that the complex ICA model is identifiable up to the trivial ambiguities and a complex linear mixture affecting those sources with identical improperness degrees [8]–[10]. However, as a direct consequence of Theorem 1, two quaternion sources with the same improperness degree [5] can be unambiguously recovered if their properness profiles are not rotationally equivalent. In other words, the quaternion ICA identifiability conditions do not only consider the improperness degree of the sources, but also the *improperness distribution*, which is measured by the properness profile. As an example, consider two sources $s_1, s_2 \in \mathbb{H}$ with properness profiles $\mathbf{\Psi}_{s_1} = [0.5i, 0, 0]^T$ and $\mathbf{\Psi}_{s_2} = [0, 0.5j, 0]^T$, which are not rotationally equivalent. Then, it is clear that s_1 and s_2 can be unambiguously recovered, even though they have identical improperness degrees [5].
- In the general quaternion case there does not exist a strong uncorrelating transform [10]. From a practical point of view, this implies that the solutions of the quaternion ICA model can not be obtained in closed form, and we have to resort to numerical algorithms. Thus, quaternion ICA can be reformulated as an approximate joint diagonalization problem [14], [15], which amounts to find the separation matrix $\mathbf{W} \simeq \mathbf{A}^{-1}$ diagonalizing the covariance and complementary covariance matrices of the observations $\mathbf{x}$, i.e.,

$$\mathbf{W}\mathbf{R}_{\mathbf{x},\mathbf{x}}\mathbf{W}^H \simeq \mathbf{I}_n, \qquad \mathbf{W}\mathbf{R}_{\mathbf{x},\mathbf{x}^{(\eta)}}\mathbf{W}^{(\eta)^H} \simeq \mathbf{\Lambda}_\eta,$$

$$\mathbf{W}\mathbf{R}_{\mathbf{x},\mathbf{x}^{(\eta')}}\mathbf{W}^{(\eta')^H} \simeq \mathbf{\Lambda}_{\eta'}, \qquad \mathbf{W}\mathbf{R}_{\mathbf{x},\mathbf{x}^{(\eta'')}}\mathbf{W}^{(\eta'')^H} \simeq \mathbf{\Lambda}_{\eta''}.$$

IV. Simulation Results

In this section, the main result of the paper is illustrated by means of a simulation example. We have considered $m = 3$ independent quaternion sources, and the entries of the square mixing matrix $\mathbf{A} \in \mathbb{H}^{3\times 3}$ have been generated as i.i.d. quaternion $\mathbb{Q}$-proper Gaussian random variables with zero mean and unit variance. The parameters of the ICA model have been estimated by means of an approximate joint-diagonalization algorithm, whose details can be found in the journal version of this paper [16]. The accuracy of the obtained results is measured by the residual mixture matrix $\mathbf{E} = \mathbf{W}\mathbf{A}$, where $\mathbf{W} \in \mathbb{H}^{3\times 3}$ is the estimated separation matrix and $\mathbf{A}$ is the actual mixing matrix. In particular, after solving the possible permutation ambiguity, the residual mixture measure for the k-th source is defined as

$$M_k = \frac{1}{|e_{k,k}|^2} \sum_{\substack{l=1 \\ l\neq k}}^{m} |e_{k,l}|^2,$$

where $e_{k,l}$ is the entry in the k-th row and l-th column of $\mathbf{E}$.

The properness profiles of the three independent sources $s_1, s_2, s_3 \in \mathbb{H}$ are

$$\boldsymbol{\psi}_{s_1} = [0.1\eta'', 0.5\eta, 0.1\eta']^T,$$

$$\boldsymbol{\psi}_{s_2} = [0.1\eta'', 0.1\eta, 0.5\eta']^T,$$

$$\boldsymbol{\psi}_{s_3} = (1-\alpha)\boldsymbol{\psi}_{s_1} + \alpha\boldsymbol{\psi}_{s_2},$$

where $0 \leq \alpha \leq 1$ is a real parameter controlling the *distances* between $\boldsymbol{\psi}_{s_3}$ and the other properness profiles. In particular, for

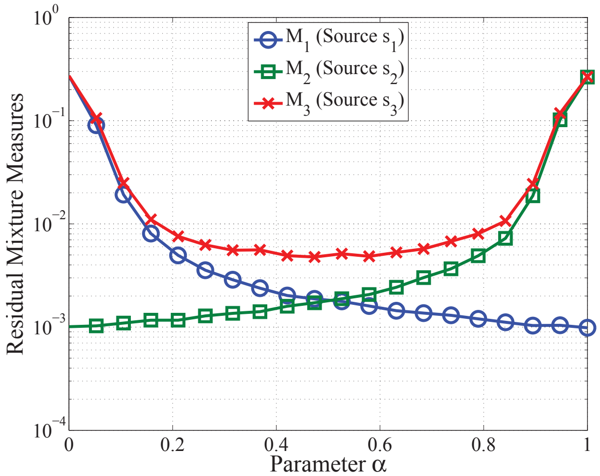

Fig. 1. Identifiability example. Three sources and 100 observations.

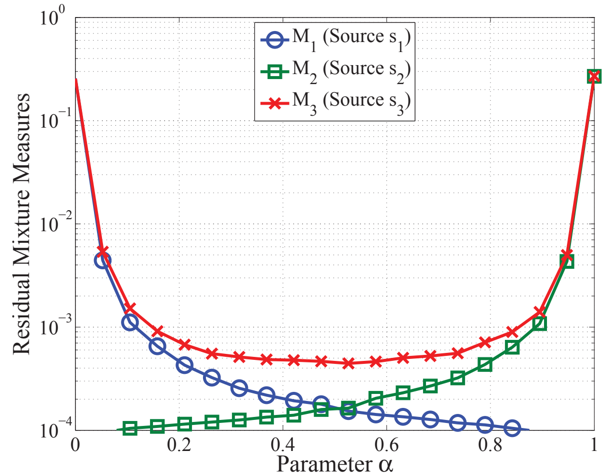

Fig. 2. Identifiability example. Three sources and 1000 observations.

$\alpha = 0$ the properness profiles $\boldsymbol{\psi}_{s_1}$ and $\boldsymbol{\psi}_{s_3}$ are rotationally equivalent, whereas $\boldsymbol{\psi}_{s_2}$ and $\boldsymbol{\psi}_{s_3}$ are rotationally equivalent for $\alpha = 1$. Furthermore, although the two first sources have the same improperness degree [5], their properness profiles are not rotationally equivalent, and therefore these sources can be unambiguously extracted.

Figs. 1 and 2 show the residual mixture measure for the three sources as a function of the parameter α. The results have been obtained by averaging 1000 independent experiments for 100 and 1000 vector observations. As stated by Theorem 1, the only non-trivial identifiability problems appear for the values of α resulting in rotationally equivalent properness profiles. Thus, there is a linear mixture of sources s_1 and s_3 for $\alpha = 0$, and a mixture of s_2 and s_3 for $\alpha = 1$. Finally, it is also interesting to note that, from a practical point of view, the accuracy of the quaternion ICA method is controlled by a tradeoff between the number of observations T and the *distances* among the different properness profiles.

V. Conclusion

In this paper we have addressed the independent component analysis (ICA) of quaternion random vectors. Specifically, we have considered the case of Gaussian data, which reduces the analysis to methods exclusively based on second-order statistics (SOS). The main result in the paper consists in the derivation of the necessary and sufficient conditions for the identifiability of the quaternion ICA model. The key role in the identifiability analysis is played by the properness profile, which is a new statistical measure for quaternion random variables, and can be seen as the quaternion counterpart of the circularity coefficients of complex vectors. Interestingly, the properness profile does not only measure the improperness degree of the sources, but also its distribution. Thus, it has been proved that the quaternion ICA model can be unambiguously recovered up to the trivial indeterminacies (permutations and quaternion scale factors) and a residual quaternion mixture involving the sources with rotationally equivalent properness profiles, i.e., properness profiles related by a three-dimensional rotation. Finally, quaternion ICA is expected to find applications in problems such as blind channel estimation in multiuser systems based on Alamouti coding.

References

[1] T. Bulow and G. Sommer, "Hypercomplex signals–a novel extension of the analytic signal to the multidimensional case," *IEEE Transactions on Signal Processing*, vol. 49, no. 11, pp. 2844–2852, Nov. 2001.

[2] N. ur Rehman and D. Mandic, "Empirical mode decomposition for trivariate signals," *IEEE Transactions on Signal Processing*, vol. 58, no. 3, pp. 1059 –1068, Mar. 2010.

[3] J. Seberry, K. Finlayson, S. Adams, T. Wysocki, T. Xia, and B. Wysocki, "The theory of quaternion orthogonal designs," *IEEE Transactions on Signal Processing*, vol. 56, no. 1, pp. 256–265, Jan. 2008.

[4] S. Sirianunpiboon, A. Calderbank, and S. Howard, "Bayesian analysis of interference cancellation for Alamouti multiplexing," *IEEE Transactions on Information Theory*, vol. 54, no. 10, pp. 4755–4761, Oct. 2008.

[5] J. Vía, D. Ramírez, and I. Santamaría, "Properness and widely linear processing of quaternion random vectors," *IEEE Transactions on Information Theory*, vol. 56, no. 7, pp. 3502 –3515, Jul. 2010.

[6] J. Vía, D. Ramírez, I. Santamaría, and L. Vielva, "Widely and semi-widely linear processing of quaternion vectors," in *IEEE Int. Conf. on Acoustics, Speech and Signal Proc. (ICASSP)*, Texas, USA, Mar. 2010.

[7] A. Hyvärinen, J. Karhunen, and E. Oja, *Independent Component Analysis*. Wiley Interscience, 2001.

[8] P. Schreier, L. Scharf, and C. Mullis, "Detection and estimation of improper complex random signals," *IEEE Transactions on Information Theory*, vol. 51, no. 1, pp. 306–312, Jan. 2005.

[9] P. Schreier, L. Scharf, and A. Hanssen, "A generalized likelihood ratio test for impropriety of complex signals," *IEEE Signal Processing Letters*, vol. 13, no. 7, pp. 433–436, July 2006.

[10] J. Eriksson and V. Koivunen, "Complex random vectors and ICA models: identifiability, uniqueness, and separability," *IEEE Transactions on Information Theory*, vol. 52, no. 3, pp. 1017–1029, Mar. 2006.

[11] J. P. Ward, *Quaternions and Cayley numbers: Algebra and applications*. Dordrecht, Netherlands: Kluwer Academic, 1997.

[12] N. N. Vakhania, "Random vectors with values in quaternion hilbert spaces," *Theory of Probability and its Applications*, vol. 43, no. 1, pp. 99–115, 1999.

[13] P. Amblard and N. Le Bihan, "On properness of quaternion valued random variables," in *IMA Conference on Mathematics in Signal Processing*, Cirencester (UK), 2004, pp. 23–26.

[14] D.-T. Pham and J.-F. Cardoso, "Blind separation of instantaneous mixtures of nonstationary sources," *IEEE Transactions on Signal Processing*, vol. 49, no. 9, pp. 1837–1848, Sep 2001.

[15] A. Ziehe, P. Laskov, G. Nolte, and K. R. Müller, "A fast algorithm for joint diagonalization with non-orthogonal transformations and its application to blind source separation," *J. Mach. Learn. Res.*, vol. 5, pp. 777–800, 2004.

[16] J. Vía, D. P. Palomar, L. Vielva, and I. Santamaría, "Quaternion ICA from second-order statistics," Submitted to *IEEE Transactions on Signal Processing*, 2010.