PARAMETRIC SMOOTHING OF SPLINE INTERPOLATION

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ABSTRACT

Cubic spline interpolation is commonly applied in signal reconstruction problems. However, overshooting between samples is normally observed, and typically the reconstructed signal does not preserve the statistical properties of the original data neither other desired properties such as monotonicity or convexity. These undesirable effects are minimized in the case of piecewise linear (PWL) interpolation, of course with a discontinuous derivative. In this paper we use a parameterized family of splines, named αsplines , that allows a smooth transition from PWL ($\alpha = 0$) to cubic spline interpolation ($\alpha = 1$). Closed-form expressions that relate α to the smoothness and variance of the interpolation are derived. Moreover, a fast interpolation technique based on digital filtering can be applied.

1. INTRODUCTION

The problem of fitting a smooth function to a given data set is commonly solved by using a cubic spline based interpolation [1]. However, in many applications this solution is not adequate, due to the large overshoots that cubic splines may undergo between data samples [2]. This overshooting also implies other undesirable features: the interpolation may not conserve the monotonicity of the data samples [3], or the statistical properties of the reconstructed signal (for instance, its power) are not consistent with the estimated variance of the original data set [4]. Of course, all these drawbacks disappear in the case of piecewise-linear (PWL) interpolation, but with a discontinuous derivative.

In this letter we use a family of spline interpolants based on B-spline convolution kernels [5] that depend on a single parameter α: when α equals zero we obtain the PWL solution, whereas if α is one we get the cubic spline interpolation. For intermediate values of the parameter we get solutions that are in between both approaches, trading off between the optimal curvature of the cubic spline interpolations and the desirable properties of the solutions closer to the PWL interpolation.

We will derive closed-form expressions for the smoothness and the variance of the interpolation as a function of α and the original data set. This expression allows seeking the value of the parameter that fits the desired solution.

In the next section, we will present the αspline kernel, its properties and the interpolation techniques. Section 3 and section 4 evaluate the smoothness and variance of the αspline interpolation, respectively, and section 5 presents some simulations and practical examples. Finally, the main conclusions are shown in section 6.

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2. αSPLINES

2.1. αspline Kernel

It is well known that a B-spline of order n can be generated by convolving $(n+1)$ times a centered normalized rectangular pulse [6, 7]. In [5] a new family of spline kernels, called spline bi-kernels, are constructed convolving two B-splines of degrees n_1, n_2 and witdths h_1, h_2. We propose in this paper a particular case of those bi-kernels and to use them in order to obtain an adecuate interpolation solution in terms of smoothness and variance.

We call αspline of parameter α, denoted by $\beta_\alpha(x)$, to the convolution of two unit-width centered normalized rectangular pulses, $\beta_0(x) = p_\alpha(x)|_{\alpha=1}$, and two α-width centered normalized rectangular pulses, $p_\alpha(x)$:

$$\beta_\alpha(x) = \beta_0(x) * p_\alpha(x) * \beta_0(x) * p_\alpha(x), \tag{1}$$

where

$$p_\alpha(x) = \begin{cases} \frac{1}{\alpha}, & |x| \le \frac{\alpha}{2}; \\ 0, & |x| > \frac{\alpha}{2}. \end{cases}$$

Hence, the following expresion for the αspline kernel is obtained:

$$\beta_\alpha(x) = \begin{cases} 1 - |x| - \frac{1}{3\alpha^2}(\alpha - |x|)^3, & x \in X_1; \\ 1 - |x|, & x \in X_2; \\ \frac{(\alpha+1)\left(3-(\alpha-2)^2-3x^2\right)}{6\alpha^2} + \frac{(\alpha-1)^2+x^2}{2\alpha^2}|x|, & x \in X_3; \\ 1 - |x| + \frac{1}{6\alpha^2}(\alpha - 1 + |x|)^3, & x \in X_4; \\ \frac{1}{6\alpha^2}(\alpha + 1 - |x|)^3, & x \in X_5; \\ 0, & x \in X_6; \end{cases}$$

where $X_1 = \{|x| < \min(\alpha, 1-\alpha)\}$, $X_2 = \{\alpha \le |x| < 1-\alpha\}$, $X_3 = \{1-\alpha \le |x| < \alpha\}$, $X_4 = \{\max(\alpha, 1-\alpha) \le |x| < 1\}$, $X_5 = \{1 \le |x| < 1+\alpha\}$, and $X_6 = \{|x| \ge 1+\alpha\}$.

Figure 1 depicts $\beta_\alpha(x)$ for different values of α. It is clear that if $\alpha = 0$ the αspline kernel is identical to B-spline of order one, $\beta_1(x)$. On the other hand, when $\alpha = 1$ the αspline kernel becomes the cubic B-spline, $\beta_3(x)$. Intermediate values of α provide base functions in between both solutions; i.e., controlling the curvature and the closeness to the linear solution. In any case, except when $\alpha = 0$, the interpolation maintains the continuity of the first and second derivatives.

2.2. αspline Interpolation

The αspline representation, $S_\alpha(x)$, of a function, $f(x)$, is

$$S_\alpha(x) = \sum_{k=-\infty}^{+\infty} c[k]\beta_\alpha(x-k), \tag{2}$$

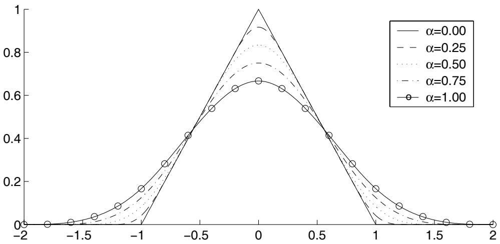

Fig. 1. αspline kernel, $\beta_\alpha(x)$, for several values of α.

where $\beta_\alpha(x-k)$ is the k-th αspline kernel, and $c[k]$ are the coefficients that meet the interpolatory condition

$$S_\alpha(k) = f(k), \quad \forall\, k \in \mathcal{Z}.$$

The value of the coefficients $c[k]$ can be evaluated in a similar way than the cubic spline interpolation coefficients [1]; i.e., by solving a tridiagonal matrix using standard numerical techniques. Then, the N point interpolation of the function $f(x)$, assuming periodical boundary conditions, requires to solve

$$\begin{bmatrix} a & b & 0 & 0 & \dots & 0 & b \\ b & a & b & 0 & \dots & 0 & 0 \\ 0 & b & a & b & \dots & 0 & 0 \\ \vdots & & & & \dots & & \vdots \\ b & 0 & 0 & 0 & \dots & b & a \end{bmatrix} \begin{bmatrix} c[1] \\ c[2] \\ c[3] \\ \vdots \\ c[N] \end{bmatrix} = \begin{bmatrix} f(1) \\ f(2) \\ f(3) \\ \vdots \\ f(N) \end{bmatrix} \tag{3}$$

where

$$\begin{aligned} a &= \beta_\alpha(x)|_{x=0} = \frac{3-\alpha}{3}, \\ b &= \beta_\alpha(x)|_{x=+1} = \beta_\alpha(x)|_{x=-1} = \frac{\alpha}{6}. \end{aligned}$$

Digital filter techniques can also be applied to solve this interpolation problem [6, 8] as Figure 2 shows. First, it is necesary to define the discrete αspline kernel, $b_\alpha[k]$,

$$b_\alpha[k] = \beta_\alpha(x)|_{x=k} = \frac{\alpha}{6}\delta[k+1] + \frac{6-2\alpha}{6}\delta[k] + \frac{\alpha}{6}\delta[k-1],$$

with z transform

$$B_\alpha(z) = \left(\alpha z + (6-2\alpha) + \alpha z^{-1}\right)/6.$$

Its inverse is given by

$$D_\alpha(z) = (B_\alpha)^{-1}(z) = \frac{-6z_1}{\alpha}\left(\frac{1}{1-z_1 z^{-1}}\right)\left(\frac{1}{1-z_1 z}\right), \tag{4}$$

with

$$z_1 = \frac{\alpha - 3 + \sqrt{9-6\alpha}}{\alpha}.$$

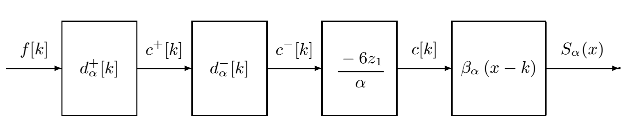

Fig. 2. αspline interpolation scheme.

Therefore, the direct filter, $d_\alpha[k]$, which provides the coefficients $c[k]$ from the function values $f(k)$, can be obtained by inverse z transform of (4)

$$d_\alpha[k] = \frac{(1-z_1)}{(z_1+1)} z_1^{|k|}, \tag{5}$$

and it can be factorized in two digital filters: one causal, $d_\alpha^+[k]$, and one anticausal, $d_\alpha^-[k]$,

$$d_\alpha^\pm[k] = z_1^{\pm k} u[\pm k].$$

Therefore, (5) can be rewritten as

$$d_\alpha[k] = \frac{-6z_1}{\alpha}\, d_\alpha^+[k] * d_\alpha^-[k].$$

Finally, the N point αspline interpolation of the function $f[k]$, with periodical constrains, can be solved in a recursive and efficient way that requires only $(5N-2)$ operations:

$$\begin{aligned} c^+[0] &= \frac{1}{1-z_1^N}\left(f[0] + \sum_{k=1}^{N-1} z_1^k f[N-k]\right); \\ c^+[k] &= f[k] + z_1 c^+[k-1], \text{ for } k = 1, 2, \dots, N-1; \\ c^-[N-1] &= \frac{1}{1-z_1^N}\left(c^+[N-1] + \sum_{k=0}^{N-2} z_1^{k+1} c^+[k]\right); \\ c^-[k] &= c^-[k+1] - z_1 c^+[k], \text{ for } k = N-2, \dots, 0; \\ c[k] &= \frac{-6z_1}{\alpha} c^-[k], \text{ for } k = 0, 1, \dots, N-1. \end{aligned}$$

3. αSPLINE INTERPOLATION SMOOTHNESS

The smoothness of a 1D function in a closed interval $[a,b]$ is defined as the inverse of the strain energy, that can be evaluated as

$$E_L = \int_a^b [f^{(2)}(x)]^2 dx.$$

where $f^{(2)}(x)$ denotes the second derivative of $f(x)$. This value is minimized by the cubic spline interpolation [1], and we will show that it grows monotonically with the smoothness parameter of the αspline interpolation. We consider the interpolation of N samples representing M periods of a sinusoidal function, $f(x)$, with discrete frequency $\omega_0 = 2\pi/T_0 = 2\pi M/N$:

$$f[k] = A\sin(\omega_0 k + \Phi_0) = A\sin\left(2\pi\frac{M}{N}k + \Phi_0\right), \tag{6}$$

where $0 \le M \le N-1$, and $0 \le \omega_0 \le 2\pi$.

Then, the strain power of the αspline interpolation can be evaluated as

$$P_S(S_\alpha(x)) = \frac{1}{MT_0}\int_{MT_0}\left(S_\alpha^{(2)}(x)\right)^2 dx. \tag{7}$$

Due to the linearity and invariance of the interpolatory process (see Figure 2) the second derivative of the αspline interpolation is

$$S_\alpha^{(2)}(x) = \sum_{k=-\infty}^{+\infty} c[k]\beta_\alpha^{(2)}(x-k) = \sum_{k=-\infty}^{+\infty} c^{(2)}[k]\frac{1}{\alpha}\Delta\left(\frac{x}{\alpha}\right),$$

where $\Delta(x)$ represents the triangular pulse function:

$$\Delta(x) = \beta_\alpha(x)|_{\alpha=0} = \begin{cases} 1-|x|, & |x|<1; \\ 0, & |x| \geq 1; \end{cases}$$

and $c^{(2)}[k]$ the discrete-time second difference of the αspline coefficients

$$c^{(2)}[k] = c[k] * h_2[k] = f[k] * d_\alpha[k] * h_2[k],$$

where $h_2 = \delta[k+1] - 2\delta[k] + \delta[k-1]$ is the second order derivative filter. Since the input signal, $f[k]$, is sinusoidal we can write

$$\begin{aligned} c^{(2)}[k] &= f[k] \cdot D_\alpha(z) \cdot H_2(z)|_{z=e^{j\omega_0}} \\ &= A\sin(\omega_0 k + \Phi_0)\frac{3}{3+\alpha(\cos\omega_0 - 1)}(2\cos\omega_0 - 2). \end{aligned}$$

Then, remembering the periodical boundary conditions, we obtain

$$P_S(S_\alpha(x)) = \frac{1}{N}\int_N \left[\sum_{k=1}^{N} c^{(2)}[k]\frac{1}{\alpha}\Delta\left(\frac{x-k}{\alpha}\right)\right]^2 dx,$$

and, after some algebra, we can express the strain power of the αspline interpolation of a sinusoidal signal as

$$P_S(S_\alpha(x)) = \frac{A^2}{2}\frac{24}{\alpha}\left(\frac{\cos\omega_0 - 1}{3+\alpha(\cos\omega_0 - 1)}\right)^2 \cdot G(\alpha, \omega_0), \quad (8)$$

where

$$G(\alpha,\omega) = \begin{cases} 1, & 0 \leq \alpha \leq 1/2; \\ 1 + \dfrac{(2\alpha-1)^3}{2\alpha^3}\cos\omega, & 1/2 < \alpha \leq 1. \end{cases}$$

It is clear from (8) that the strain power of a unit power sinusoidal signal can be evaluated as a function of its frequency, ω, and the smoothness parameter, α, using

$$P_S(\alpha,\omega) = \frac{24}{\alpha}\left(\frac{\cos\omega - 1}{3+\alpha(\cos\omega - 1)}\right)^2 \cdot G(\alpha,\omega). \quad (9)$$

Now we consider the αspline interpolation of an arbitrary secuence, $f[k]$, of length N, which discrete Fourier transform is

$$F[n] = \sum_{k=0}^{N-1} f[k]e^{-j2\pi nk/N},$$

where $\omega = 2\pi n/N$ is the discrete frequency. Due to the orthogonalty of the sinusoidal terms, the strain power can be evaluated as

$$P_S = \frac{1}{N^2}\sum_{n=0}^{N-1} |F[n]|^2 P_S(\alpha, \omega = 2\pi n/N) \quad (10)$$

4. αSPLINE INTERPOLATION VARIANCE

The variance of the αspline interpolation of a sinusoidal function, (6), can be evaluated using (2), as

$$\begin{aligned} \mathrm{Var}(S_\alpha(x)) &= \frac{1}{MT_0}\int_{MT_0} (S_\alpha(x))^2 dx \\ &= \frac{1}{N}\int_N \left(\sum_{k=1}^{N} c[k]\beta_\alpha(x-k)\right)^2 dx. \end{aligned}$$

If we take into account the compact support of the base function, $\beta_\alpha(x)$, and the periodical constrains, we obtain the variance as a function of α and of the discrete frequency, ω,

$$\mathrm{Var}(\alpha,\omega) = 9\frac{I_0 + 2I_1\cos\omega + 2I_2\cos 2\omega + 2I_3\cos 3\omega}{[3+\alpha(\cos\omega - 1)]^2}, \quad (11)$$

where

$$I_k(\alpha) = \int_{-\infty}^{\infty} \beta_\alpha(x)\beta_\alpha(x-k)\,dx$$

are polinomial functions of the parameter α.

Then, the variance of the αspline interpolation of an arbitray secuence, $f[k]$, can be evaluated as

$$Var = \frac{1}{N^2}\sum_{n=1}^{N-1} |F[n]|^2 \mathrm{Var}(\alpha, \omega = 2\pi n/N). \quad (12)$$

5. SIMULATIONS AND APPLICATION EXAMPLES

First, we have verified the validity of the smoothness (10) and variance (12) expressions by means of simulations. As an example, we generate $N = 512$ samples of a zero-mean Gaussian signal, $f[k]$, and perform the αspline interpolation for two distinct smoothness parameter values ($\alpha_1 = 0.3$, and $\alpha_2 = 0.8$). To estimate the smoothness and the variance of the continuous-time signal, we use a resampling factor of $R = 1000$; i.e., obtaining sequences of $L = N \cdot R = 512000$ samples. Figure 3 shows a section of the interpolated signals and their respective second derivatives. It becomes clear the greater smoothness of the interpolation with $\alpha = 0.8$ and the larger values of the second derivative of the interpolation with $\alpha = 0.3$.

In a quantitative way, the variance and the strain power values of the interpolated signals (evaluated as the quadratic mean of the samples of the signal and of their second derivative respectively) exactly match up with the values obtained from the analytical expressions; as Table 1 shows.

From (11), it can be stated that the variance of the αspline interpolation monotonically increases with α for any value of the frequency ω, except for a small region ($0.7 < \alpha < 1$, and $\omega > 0.9\pi$)

		$\alpha = 0.3$	$\alpha = 0.8$
Signal Variance	using (12)	0.782817	0.867299
	simulated	0.782817	0.867299
Strain Power	using (10)	17.46541	13.32091
	simulated	17.46551	13.32093

Table 1. Analytical and simulated values of the variance and strain power for two values of α.

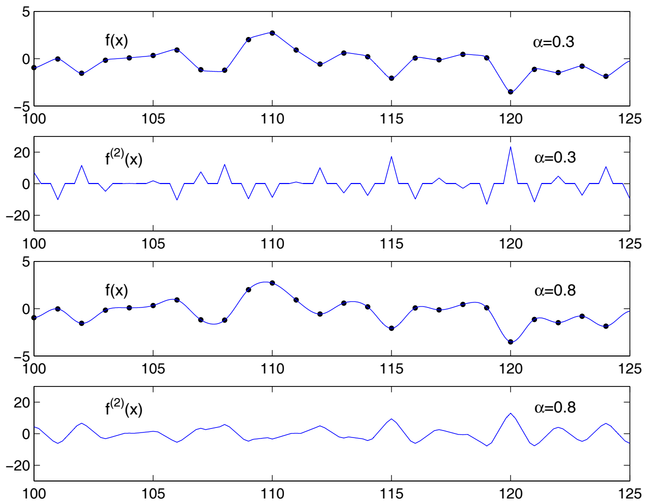

Fig. 3. αspline interpolation and its second derivative for $\alpha = 0.3$ and $\alpha = 0.8$.

where the variance is nearly constant. Similarly, from (9) we can assert that the αspline interpolation smoothness increases (the power strain decreases) with the parameter α for every ω. Again, in a small region ($0.5 < \alpha < 0.65$, and $\omega > 0.9\pi$) the behaviour is nearly constant. Therefore, in a practical situation where the signal to interpolate is formed by multiple frequency components, the variance and the smoothness of the αspline interpolation increases as α increases. This fact allows us to look for the value of α that matches the desired variance or smoothness of the interpolated signal prior to performing the interpolation. We can obtain the value α_0 that produces the desired power strain, P_{S_0}

$$P_S(\alpha_0) = P_{S_0},$$

using standard numerical methods. Similar techniques can be used to obtain the desired variance.

It is possible to use αsplines to interpolate non stationary signals and to control the variance or the smoothness of the solution using distinct values of the parameter α in distinct regions of the signal. This can be accomplished by applying

$$S_\alpha(x) = \sum_{k=-\infty}^{+\infty} c[k]\beta_{\alpha_k}(x-k),$$

and substituting every element (i,k) of the matrix in (3) with the value $\beta_{\alpha_k}(i-k)$. An example of this application is shown in Figure 4, where three different values of α have been used to control the highest value of the second derivative of the αspline interpolation despite the nonstationary variance of the original samples.

6. CONCLUSIONS

A parameterized family of splines that allows a smooth transition from the PWL to the cubic spline interpolations has been presented, and analytical expressions for the smoothness and the variance of the interpolation have been derived and validated through simulations. The αspline interpolation problem can be solved using the standard matrix formulation or digital filter techniques (with lower computational and memory cost).

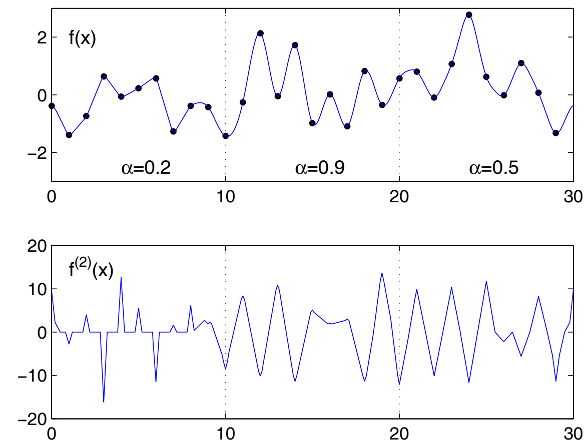

Fig. 4. Multiple α interpolation.

In applications where the variance or the smoothness is an important issue, the αspline interpolation can play an important role by taking advantage of the a priori control over both characteristics.

The extension of the αspline interpolation to a nonuniform grid is straightforward, but some work must be still carried out. Finally, there are some values of the smoothness parameter of the αsplines that produce digital filters with roots of the form $z_1 = 2^{-n}$, allowing a fast hardware implementation of the interpolation filters by means of simple bit shifts instead of multiplications.

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