INTERFERENCE ALIGNMENT IN SINGLE-BEAM MIMO NETWORKS VIA HOMOTOPY CONTINUATION

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ABSTRACT

In this paper we consider the application of a homotopy-continuation based method for finding interference alignment (IA) solutions for the deterministic K-user multiple-input multiple-output (MIMO) channel, when all users wish to send one stream of data. Homotopy continuation is based on the idea of deforming a start system, whose solution can easily be found, to reach the target system that we want to solve. For the IA problem we show that a good initial system is obtained by considering a rank-one approximation of the original MIMO interference channels. Specifically, as long as the original system is feasible, a rank-one approximation of the MIMO channels allow us to find a closed-form interference-free solution. The proposed algorithm is shown to have a lower complexity than previous methods with comparable sum-rate performance. Furthermore, it is also shown that the trivial system (rank-one MIMO channels) and target system (full-rank MIMO channels) have exactly the same number of solutions. Exploiting this equivalence, an efficient method to enumerate all the IA solutions that exist in a single-beam MIMO network is proposed.

Index Terms— Interference alignment, interference channel, MIMO, homotopy continuation, mixed volume.

1. INTRODUCTION

The degrees of freedom (DoF) of wireless interference networks represent the number of non-interfering data streams that can be simultaneously transmitted over the network. Recently, it has been shown that to achieve all or most of the achievable DoF of K-user multiple-input multiple-output (MIMO) networks with constant channel coefficients, the interference from other transmitters must be aligned at each receiver in a lower-dimensional subspace [1]. This is the basic idea of the interference alignment (IA) technique.

In this paper we focus on single-beam K-user MIMO networks in which the transmitters use only one beamforming vector. Furthermore, the decoders are assumed to be linear. Even for single-beam networks, closed-form IA solution are only known so far for some particular scenarios, such as the well-known 3-user channel with 2 antennas at both sides of the link, or the 4-user MIMO network with 2 and 3 antennas at the transmitter and receiver side, respectively [2]. Also, a constructive method for finding closed-form solutions has been proposed for the case where the number of transmit antennas equals the number of receive antennas ($N_T = N_R = N$) and the number of users equals $K = N + 1$ [3].

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In more complex scenarios, we typically have to resort to iterative algorithms to find the set of IA beamformers. To this end, a popular algorithm based on an alternating minimization procedure was proposed in [4, 5]. Other algorithms that do not attempt perfect alignment and consider alternative optimization criteria such as the minimum mean-square error (Min MSE) [6], or the maximum signal-to-interference noise ratio (Max SINR) [4] have been also considered. However, the computational complexity of all these approaches becomes a problem in practice.

In this paper, we explore the application of a homotopy-continuation based method for solving the multivariate polynomial system of equations that results from the interference alignment conditions. Homotopy methods find the solution of a target system of nonlinear equations by smoothly deforming the known solutions of a start system (typically a system with a trivial solution). For the IA problem we show that a good initial system is obtained by considering a rank-one approximation of the original MIMO interference channels. Specifically, as long as the original system is feasible, a rank-one approximation of the MIMO channels allows us to find an interference-free solution in which each transmitter employs zero-forcing beamforming (ZF-BF) to a subset of receivers. Interestingly, exploiting classical results in algebraic geometry [7] we also show that the number of solutions of the trivial system (using rank-one channels) is exactly the same as the number of IA solutions of the original system with full-rank channels. Since the rank-deficient system can be trivially solved, this allow us to count and find (via homotopy-continuation) all IA solutions for a given problem.

2. SYSTEM MODEL AND BACKGROUND

We consider the K-user interference channel, comprised of K transmitter - receiver pairs (links) that interfere with each other. We assume that all users wish to send one stream of data and are equipped with N_T and N_R antennas at each side of the link. These scenarios are usually denoted as $(N_T \times N_R, 1)^K$. Let now $\mathbf{v}_k \in \mathbb{C}^{N_T \times 1}$ be the transmit beamforming vector for user k. The discrete-time signal at receiver k is the superposition of the signals transmitted by the K users, weighted by their respective channel gains and affected by noise, i.e.,

$$\mathbf{y}_k = \mathbf{H}_{kk}\mathbf{v}_k s_k + \sum_{l \neq k} \mathbf{H}_{kl}\mathbf{v}_l s_l + \mathbf{w}_k, \qquad (1)$$

where $\mathbf{H}_{kl} \in \mathbb{C}^{N_R \times N_T}$ is the flat-fading MIMO channel from transmitter l to receiver k, $s_l \in \mathbb{C}$ is the signal transmitted by the l-th user, and $\mathbf{w}_k$ is the additive and spatially white Gaussian noise at receiver k.

For the MIMO network in (1), interference alignment is possible if there exists a set of unit-norm beamformers $\mathbf{v}_l$ ($l = 1, \ldots, K$) and unit-norm decoders $\mathbf{u}_k$ $(k = 1, \dots, K)$ such that

$$\mathbf{u}_k^H \mathbf{H}_{kl} \mathbf{v}_l = 0, \qquad \forall k \neq l \qquad \text{and} \tag{2}$$

$$\mathbf{u}_k^H \mathbf{H}_{kk} \mathbf{v}_k \neq 0. \tag{3}$$

The Tx and Rx beamformers in (2) are rotationally invariant. Therefore, without loss of generality we can assume that the first element of each vector is fixed to one (e. g. $\mathbf{v}_l = [1\ v_l^{(1)}\ \dots\ v_l^{(N_T-1)}]^T$). Moreover, for notational convenience we assume that the entries of the Rx beamformer are complex conjugated, $\mathbf{u}_k = [1\ u_k^{(1)\dagger}\ \dots\ u_k^{(N_R-1)\dagger}]^T$, where complex conjugation is denoted with the † operator.

From (2) it is evident that finding an IA solution amounts to solving a system, $\mathcal{E}$, of bilinear equations, where each equation has $(N_T - 1) + (N_R - 1)$ free variables [2]. For example, for the $(3 \times 3, 1)^5$ MIMO network (for which no closed-form IA solution is known) the structure of one of these bilinear equations is

$$\begin{aligned} &h_{kl}^{(1,1)} + h_{kl}^{(1,2)} v_l^{(1)} + h_{kl}^{(1,3)} v_l^{(2)} + \\ &h_{kl}^{(2,1)} u_k^{(1)} + h_{kl}^{(2,2)} v_l^{(1)} u_k^{(1)} + h_{kl}^{(2,3)} v_l^{(2)} u_k^{(1)} + \\ &h_{kl}^{(3,1)} u_k^{(2)} + h_{kl}^{(3,2)} v_l^{(1)} u_k^{(2)} + h_{kl}^{(3,3)} v_l^{(2)} u_k^{(2)} = 0 \end{aligned} \tag{4}$$

where $h_{kl}^{(i,j)}$ is the ij-th entry of $\mathbf{H}_{kl}$.

The solvability of the multivariate polynomial system $\mathcal{E}$ has been analyzed in [2], where it has been shown that the $(3 \times 3, 1)^5$ system is solvable because it consists of $N_e = K(K-1) = 20$ equations in the same number of variables $(N_v = K(N_T + N_R - 2) = 20)$. Moreover, the system is generic [8, Chapter 4] due to the assumption of MIMO channel matrices with independent and identically distributed (i.i.d.) elements. To solve this multivariate polynomial system, one can resort to iterative approaches such as as those proposed in [4] and [5]. In these techniques, the precoders $\mathbf{v}_l$ are first fixed to some random value thus reducing $\mathcal{E}$ to a system of linear equations in $\mathbf{u}_k$, which can be easily solved. In the next step, the values of $\mathbf{u}_k$ are fixed and a solution to $\mathbf{v}_l$ is found by solving again a linear system. The process is iterated until convergence.

Another well-known method in the mathematical literature for numerically solving systems of polynomial equations is the homotopy continuation technique [8, 9]. The main contribution of this paper is to particularize this mathematical tool for the system $\mathcal{E}$ resulting from the IA conditions, and discuss the advantages of this approach.

3. A HOMOTOPY CONTINUATION METHOD FOR IA

Homotopy continuation is a method for solving systems of polynomial equations which is based on the idea of defining a parametrized transformation that gradually deforms a start system, whose solutions are known, to reach the unknown solutions of the system that we wish to solve, or target system. Usually, the start system is chosen to be easily solvable, therefore, it is called trivial system [8, 9].

3.1. Closed-form solution for rank-one channels

A suitable trivial system for our problem can be found by considering rank-one MIMO interference channels, $\bar{\mathbf{H}}_{kl}$, which can be obtained by truncating the singular value decomposition (SVD) of the original full-rank channel matrices, $\mathbf{H}_{kl}$ [1]. The main point of the proposed rank-one approximation is that the bilinear equations in the target system, $\mathcal{E}$, can be factored as a product of two linear terms. The first one is a linear combination of the decoder variables, whereas the second term only involves the precoder variables. More precisely, if the rank-one MIMO channel $\bar{\mathbf{H}}_{kl}$ is given by

$$\bar{\mathbf{H}}_{kl} = \sigma_{kl} \mathbf{f}_{kl} \mathbf{g}_{kl}^H \tag{5}$$

the bilinear equations obtained from replacing $\mathbf{H}_{kl}$ by $\bar{\mathbf{H}}_{kl}$ in (2) can be written in the following form

$$\underbrace{\mathbf{u}_k^H \mathbf{f}_{kl}}_{L(\mathbf{u}_k)} \underbrace{\mathbf{g}_{kl}^H \mathbf{v}_l}_{L(\mathbf{v}_l)} = 0, \tag{6}$$

where $L(\mathbf{u}_k)$ is a linear form in the variables $u_k^{(i)}$, $\forall i \in \{1, \dots, N_R - 1\}$, while $L(\mathbf{v}_l)$ is a linear form in the precoder variables $v_l^{(j)}$, $\forall j \in \{1, \dots, N_T - 1\}$. Taking again the $(3 \times 3, 1)^5$ scenario to illustrate this point, one of the bilinear equations for the rank-one approximation would be

$$(f_{kl}^{(1)} + f_{kl}^{(2)} u_k^{(1)} + f_{kl}^{(3)} u_k^{(2)})(g_{kl}^{(1)\dagger} + g_{kl}^{(2)\dagger} v_l^{(1)} + g_{kl}^{(3)\dagger} v_l^{(2)}) = 0.$$

The new set of bilinear equations, denoted here as $\bar{\mathcal{E}}$, can be trivially solved by nulling a subset of linear terms in such a way that all the nulled linear factors form a full-rank linear system. In other words, a valid subset of linear terms must be a linear system with as many variables as independent equations. Thus, solving the trivial system consists of solving a linear equation system. Looking more carefully at the IA conditions (2) for rank-one channels, we observe that strict interference alignment in this case is neither needed nor possible. Actually, with rank-one MIMO interference channels it is possible to create, at any given receiver, a subspace free of interference of the required dimensionality just by selecting a subset of interfering users that employ zero-forcing beamforming (ZF-BF) for that particular receiver. In other words, each transmitter must design its signal to lie in the nullspace of a chosen rank-one MIMO channel. This idea is depicted in Fig. 1 for the $(3 \times 3, 1)^5$ network. Specifically, users 4 and 5 employ ZF-BF so as to not cause interference on user 1, whereas the interference caused by users 2 and 3 spans a subspace

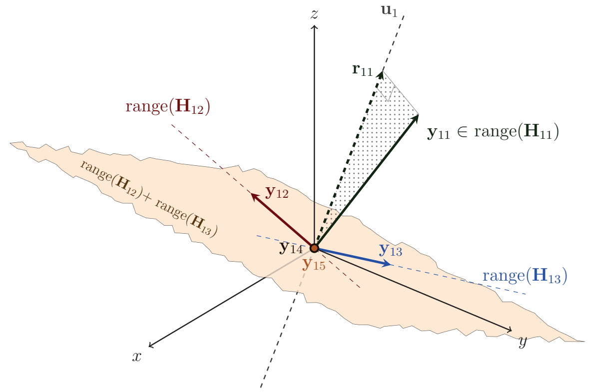

Fig. 1. Signal space at the receiver of user 1 for the $(3 \times 3, 1)^5$ scenario with rank-one MIMO channels. In this case, user 1 only sees the interference coming from users 2 and 3 (users 4 and 5 employ ZF-BF), which spans a subspace of dimension 2. The direction orthogonal to that subspace provides the required signalling dimension free of interference ($\mathbf{u}_1$).

[1] It is interesting to remark that rank-one MIMO channels also appear in single-input multiple-output (SIMO) interference networks when symbol extensions are applied over time-invariant channels, or due to the propagation geometry ("keyhole channel").

of dimension 2 (remember that we are considering rank-one channels). Therefore, user 1, who observes a 3-dimensional subspace, has one signalling dimension free of interference ($\mathbf{u}_1$ in the figure). Let us notice, finally, that global channel knowledge is not required anymore in order to obtain interference-free signalling dimensions with rank-one channels. Nevertheless, certain coordination among users is still needed in order to determine which subset of interfering users should employ ZF-BF at a given Rx.

3.2. Path-following procedure

The general IA problem in (2) can be reliable solved by defining a simple homotopy $\mathbf{q}$ as follows

$$\mathbf{q}(\mathbf{x},t) = \gamma(1-t)\bar{\mathcal{E}}(\mathbf{x}) + t\mathcal{E}(\mathbf{x}), \qquad \gamma \in \mathbb{C}, \quad t \in [0,1] \quad (7)$$

where $\bar{\mathcal{E}}$ is the start system and $\mathcal{E}$ is the target system. The homotopy $\mathbf{q}$ is a $N_v \times 1$ column vector and is a function of the vector $\mathbf{x}$, and the continuation parameter, t. The vector $\mathbf{x}$ is a $N_v \times 1$ vector that contains all the variables in $\bar{\mathcal{E}}$ or $\mathcal{E}$. The constant γ is randomly chosen to guarantee that the solution paths defined by the homotopy are regular. A solution path is regular if the Jacobian matrix[2] of $\mathbf{q}$ is regular for all $t \in [0,1)$.

Our goal is to move along the path, from $t = 0$ to $t = 1$ in small steps, Δt. Usually, this path is followed by using a numerical predictor/corrector method. Basic Euler prediction and Newton correction, are both accomplished by considering a local model of the homotopy function via its first order approximation:

$$\mathbf{q}(\mathbf{x}+\Delta\mathbf{x}, t+\Delta t) \simeq \mathbf{q}(\mathbf{x},t) + \mathbf{Q}_x(\mathbf{x},t)\Delta\mathbf{x} + \mathbf{q}_t(\mathbf{x},t)\Delta t \quad (8)$$

where $\mathbf{Q}_{\mathbf{x}} = \frac{\partial \mathbf{q}}{\partial \mathbf{x}}$ is the $N_v \times N_v$ Jacobian matrix and $\mathbf{q}_t = \frac{\partial \mathbf{q}}{\partial t}$ has size $N_v \times 1$. If we have a point $(\mathbf{x}_1, t_1)$ near the path, that is, $\mathbf{q}(\mathbf{x}_1, t_1) \simeq 0$, one may predict to a new approximate solution at $t_1 + \Delta t$ by setting $\mathbf{q}(\mathbf{x} + \Delta\mathbf{x}_p, t_1 + \Delta t) = 0$ and solving the resulting linear system to get the Euler prediction step

$$\Delta\mathbf{x}_p = -\mathbf{Q}_{\mathbf{x}}^{-1}(\mathbf{x}_1, t_1)\mathbf{q}_t(\mathbf{x}_1, t_1)\Delta t.$$

On the other hand, when $\mathbf{q}(\mathbf{x}_1, t_1)$ is not as small as one would like, one may hold t constant by setting $\Delta t = 0$ and solving the equation to get the Newton correction step

$$\Delta\mathbf{x}_c = -\mathbf{Q}_{\mathbf{x}}^{-1}(\mathbf{x}_1, t_1)\mathbf{q}(\mathbf{x}_1, t_1).$$

Although these are the conventional steps of a path-following procedure, there are many possible choices for the implementation of each step. A common strategy is to execute the Newton correction step for a maximum of `MaxNwtIter` times. The correction is said to be successful if Newton's method converges within a pre-specified path-tracking tolerance, `NwtTol`, within the allowed number of iterations. Usually, a good choice of the parameters is: `MaxNwtIter` $= 3$ and `NwtTol` $= 10^{-4}$. Also, the step length must be adapted depending on the success of the path tracking. This path-following procedure is carried out in the Step 4 of the overall algorithm that is summarized in Algorithm 1.

4. NUMBER OF IA SOLUTIONS

The equivalence between the IA problem and the solution of a set of multivariate polynomial equations has been exploited in [2] to find how many different IA solutions exist for a particular scenario (or to provide upper bounds when the exact number of solutions cannot be found). Based on classic results from algebraic geometry, in [2] it is stated that, for feasible single-beam MIMO networks, the number of IA solutions coincides with the mixed volume of the Newton polytopes that support each equation of the system (i.e., $\mathcal{N}(\mathcal{E}) = \mathcal{MV}(\mathcal{E})$). Although this solves theoretically the problem, at least for single-beam networks, in practice the computation of the mixed volume of a set of bilinear equations using the available software tools [10] can be very demanding. In consequence, the exact number of IA solutions was only known so far for some particular cases [2, 11].

Interestingly, the introduction of rank-one MIMO channels in the network also opens the possibility to count the exact number of IA solutions in a much more efficient way. The main point is to notice that the number of solutions of the trivial system coincides with the number of solutions of the original system with full-rank channels, $\mathcal{N}(\bar{\mathcal{E}}) = \mathcal{N}(\mathcal{E})$. The proof is straightforward by taking into account that the same monomials are present in both $\bar{\mathcal{E}}$ and $\mathcal{E}$. We can now use a tree search approach along with a backtracking procedure to count the number of solutions of $\mathcal{N}(\bar{\mathcal{E}})$ [3]. Since we are exploiting the specific structure of the IA bilinear equations, this

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Algorithm 1: Perfect interference alignment via homotopy continuation for single-beam scenarios.

1. Obtain the best rank-one approximation, $\bar{\mathbf{H}}_{kl}$, of every channel matrix $\mathbf{H}_{kl}$ via the SVD decomposition.
2. Build the start system, $\bar{\mathcal{E}}(\mathbf{x}) = 0$, from channel matrices $\bar{\mathbf{H}}_{kl}$ and the target system, $\mathcal{E}(\mathbf{x}) = 0$, from channel matrices $\mathbf{H}_{kl}$.
3. Obtain one solution, $\mathbf{x}_{\bar{\mathcal{E}}}$, to the start system.
4. Execute the path-following procedure on the homotopy

$$\mathbf{q}(\mathbf{x},t) = \gamma(1-t)\bar{\mathcal{E}}(\mathbf{x}) + t\mathcal{E}(\mathbf{x})$$

Input: $\mathbf{q}(\mathbf{x},t)$, $\mathbf{x}_{\bar{\mathcal{E}}}$, Δt, `NwtTol`, `MaxNwtIter`
Output: A solution to the target system, $\mathbf{x}_{\mathcal{E}}$
$t = 0, \mathbf{x} = \mathbf{x}_{\bar{\mathcal{E}}}$
$t^* = t, \mathbf{x}^* = \mathbf{x}$
while $t < 1$ **do**
  `// Euler prediction`
  $\Delta\mathbf{x}_p = -\mathbf{Q}_{\mathbf{x}}^{-1}(\mathbf{x},t)\mathbf{q}_t(\mathbf{x},t)\Delta t$
  $\mathbf{x} = \mathbf{x} + \Delta\mathbf{x}_p$
  `// Newton correction`
  $t = t + \Delta t$
  for *iter=1* **to** *MaxNwtIter* **do**
    $\Delta\mathbf{x}_c = -\mathbf{Q}_{\mathbf{x}}^{-1}(\mathbf{x},t)\mathbf{q}(\mathbf{x},t)$
    $\mathbf{x} = \mathbf{x} + \Delta\mathbf{x}_c$
    if $||\Delta\mathbf{x}_c|| <$ *NwtTol* **then**
      Success $=$ `true` **break**
  if *Success* **then**
    $\Delta t = 2\Delta t$
    $t^* = t, \mathbf{x}^* = \mathbf{x}$
  else
    $\Delta t = \Delta t/2$
    $t = t^*, \mathbf{x} = \mathbf{x}^*$
$\mathbf{x}_{\mathcal{E}} = \mathbf{x}$

5. Build precoders, $\mathbf{v}_l$, and decoders, $\mathbf{u}_k$, from the solution $\mathbf{x}_{\mathcal{E}}$.

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[2] The Jacobian matrix $\mathbf{F}_{\mathbf{x}} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}$ of a system $\mathbf{f}$ is a matrix whose entries are the derivatives of each equation with respect to the variables in $\mathbf{f}$.

[3] Due to the lack of space we do not provide details here.

Scenario	$(2\times(K-1),1)^K$	$(3\times(K-2),1)^K$	$(4\times(K-3),1)^K$
K=3	2	–	–
K=4	9	9	–
K=5	44	216	44
K=6	265	7570	7570
K=7	1854	357435	1975560
K=8	14833	22040361	749649145

Table 1. Number of IA solutions for several symmetric single-beam scenarios.

procedure is much more efficient than resorting to general software packages to compute the mixed volume. Using this method we have obtained the results in Table 1. As an example, our enumeration method for the $(3\times5,1)^7$ scenario takes less than 6 minutes to count the 357435 solutions, whereas the existing software takes several days to obtain the same solution.

5. NUMERICAL RESULTS

In this section, the proposed algorithm is evaluated in terms of sum-rate performance and computational complexity. We consider the $(3\times3,1)^5$ interference MIMO channel and evaluate the average sum-rate with respect to the SNR for the proposed IA algorithm (`IA-HC`), the conventional IA (`IA-AM`) and `MaxSINR` algorithms in [4]. The results of 100 independent realizations were averaged. In Fig. 2 we observe that the proposed algorithm provides a sum-rate performance comparable to that provided by the `IA-AM` algorithm. However, one of the advantages of the proposed algorithm is that it can find and trace all the IA solutions starting from a simplified network created from rank-one channels. For this setting the total number of solutions is 216. With the proposed method, it is very easy to find all the different solutions for the rank-one channels, and then, apply the homotopy-continuation method to get all the solutions for the full-rank channels. This would be impossible for the conventional IA algorithm, which typically starts the alternating minimization procedure with a set of random precoders. Therefore, with the proposed method it is also possible to find the maximum sum-rate solution (`IA-MaxSR-HC` in the figure) by exhaustive search on the whole set of solutions. As expected, for low SNRs the best results are provided by the `MaxSINR` algorithm, but at high SNRs, where a perfectly aligned solution is desirable, the `IA-MaxSR-HC` provides the best results.

Finally, although we do not have enough space to provide a detailed analysis, let us mention that the computational complexity of the `IA-HC` algorithm is lower than that of the `IA-AM` algorithm. Just to give an idea, for the $(3\times3,1)^5$ scenario, the total number of complex floating point operations required to compute a solution with the `IA-HC` algorithm is approximately five times lower than with the conventional IA algorithm.

6. CONCLUSION

In this paper we have proposed a new method for solving the single-beam IA problem via homotopy continuation. The basic idea is deforming a start system, whose solution can easily be found, to reach the target system that we want to solve. We have shown that a rank-one approximation of the original MIMO interference channels gives a good start system and allows to enumerate all the IA solutions. The proposed algorithm is shown to have a lower complexity than previous methods with comparable sum-rate performance. As a further line, we are studying how to extend the proposed technique to multi-beam scenarios. In this case, the start system should be obtained from rank-d channels, where d is the number of interference-free streams per user.

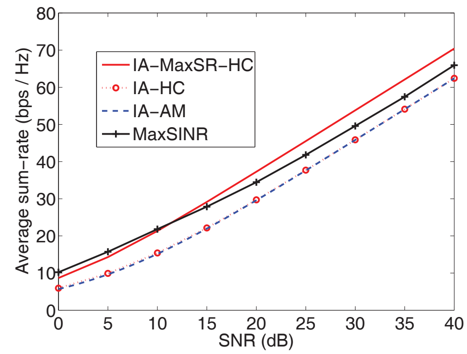

Fig. 2. Average sum-rate performance of the `IA-AM`, `MaxSINR`, `IA-HC` and `IA-MaxSR-HC` algorithms for the $(3\times3,1)^5$ scenario.

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