Approximate closed-form expression for the ergodic capacity of polarisation-diversity MIMO systems

J. Pérez, J. Ibáñez, L. Vielva and I. Santamaría

MIMO systems using single dual-polarised antennas at transmitter and receiver can be a simple, cheap and compact alternative to conventional multi-antenna MIMO configurations. Recently, approximate expressions and bounds for the ergodic capacity of such systems have been proposed assuming Rayleigh or Ricean channel models. A tight closed-form approximation for the ergodic capacity of such systems in arbitrary multipath fading channels is derived.

Introduction: Conventional multiple-input multiple-output (MIMO) systems use multiple antennas at both ends of the wireless link. To exploit the potential benefits of such systems, the spacing between antennas must be large enough to avoid high spatial correlation. But in small micro- and pico-cells, the available space is limited. A simple, cheap and compact alternative is to employ single dual-polarised antennas at transmitter (Tx) and receiver (Rx). In this case, using a single antenna at both ends, a 2×2 MIMO configuration is obtained by employing two orthogonal polarisations. This Letter focuses on such systems.

Analytic bounds and approximations for the ergodic capacity of MIMO systems can be found in the technical literature (see for example [1]). They assume non-physical channel models as Rayleigh or Ricean fading channels. In this Letter we use a multipath physical channel model to derive a tight closed-form approximation for the ergodic capacity of 2×2 MIMO systems based on dual-polarised antennas. This expression depends on the specific multipath structure of the channel, and encompasses the Rayleigh and Ricean fading channels as particular cases.

MIMO channel model: The channel is assumed to be flat over the frequency band and quasistatic. The MIMO channel is modelled using the so-called USLAC (uncorrelated stochastic local area channel) model [2]. We assume that there is no spatial diversity between the polarisation branches of each antenna, i.e. the separation between the phase centres associated with each polarisation is much less that of the carrier wavelength. Then the baseband equivalent MIMO channel matrix can be expressed as follows:

$$\boldsymbol{H} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} = \sum_{n=1}^{N} \exp(j\phi_n) \begin{bmatrix} V_{11}^n & V_{12}^n \\ V_{21}^n & V_{22}^n \end{bmatrix} \quad (1)$$

where diagonal elements of $\boldsymbol{H}$ correspond to transmission and reception on the same polarisation, while the off-diagonal entries correspond to transmission and reception on orthogonal polarisations. The entries of $\boldsymbol{H}$ are modelled as the sum of the contributions of N uniform plane waves reaching the Rx, due to the multipath propagation. The multipath terms V_n^{ij} are complex values that depend on the antenna polarisation vectors (one polarisation vector for each polarisation branch) and on the specific propagation scenario through reflections, diffractions and scattering of the multipath waves in the environment. According to the USLAC model the multipath phases are considered uncorrelated uniformly distributed random variables, each one associated with a multipath wave. The Rayleigh and Ricean fading channels are encompassed as particular cases of this channel model [2]. All the MIMO channel characteristics, like the imbalances and correlation between the matrix elements, are determined by the values of the multipath terms.

Ergodic capacity: We assume that the channel matrix is known at the Rx, but unknown at the Tx. Then, for a given realisation of the channel $\boldsymbol{H}$, the capacity (in bit/s/Hz) can be expressed as follows [3]:

$$C = \log_2 \det\left(\boldsymbol{I}_2 + \frac{E_s}{2\sigma^2}\boldsymbol{Q}\right), \quad \boldsymbol{Q} = \boldsymbol{H}^H\boldsymbol{H} \quad (2)$$

where $\boldsymbol{I}_2$ is the 2×2 identity matrix, E_s is the total average energy transmitted over a symbol interval, σ^2 is the average noise power at the polarisation branches of the Rx antenna and the superscript $(\cdot)^H$ represents the Hermitian operator. The ergodic capacity is defined as the ensemble average of the capacity for all of the channel realisations: $E[C]$.

To focus on the effect of multipath fading only, we normalise the channel matrix realisations by removing the average path-loss (α):

$$\boldsymbol{H}_{norm} = \frac{\boldsymbol{H}}{\sqrt{\alpha}}, \quad \alpha = \frac{1}{4}E\left[\sum_{i=1}^{2}\sum_{j=1}^{2}|h_{ij}|^2\right] = \frac{1}{4}\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{n=1}^{N}|V_n^{ij}|^2 \quad (3)$$

After the channel normalisation, the average received power at each polarisation branch equals the total transmitted power, so the average signal-to-noise ratio will be $\rho = E_s/\sigma^2$.

To obtain an analytical expression of the ergodic capacity, we first expand (2) in Taylor series about the expectation of the determinant, and then apply the expectation operator. The resulting expression is

$$E[C] = \log_2 E[D] - \frac{\log_2(e)}{2}\frac{E[D^2] - (E[D])^2}{(E[D])^2} + \cdots,$$
$$D = \det\left(I_2 + \frac{\rho}{2}\boldsymbol{Q}\right) \quad (4)$$

The first two moments of D can be expressed analytically as a function of the elements of $\boldsymbol{Q}$

$$E[D] = 1 + 2\rho\alpha + \frac{\rho^2}{4}E[q_{11}q_{22} - q_{12}q_{21}] \quad (5)$$

$$E[D^2] = 1 + 4\rho\alpha + \frac{\rho^2}{4}E[q_{11}^2 + q_{22}^2 + 4q_{11}q_{22} - 2q_{12}q_{21}] + \frac{\rho^3}{4}E[q_{11}^2 q_{22} + q_{11}q_{22}^2 - q_{12}q_{21}(q_{11} + q_{22})] + \frac{\rho^4}{16}E[q_{11}^2 q_{22}^2 + q_{12}^2 q_{21}^2 - 2q_{12}q_{21}q_{11}q_{22}] \quad (6)$$

Considering the statistical distribution of the multipath phases, the expectation of the powers of the q_{ij} terms can be obtained analytically as follows:

$$E[q_{ij}q_{ks}] = \sum_{n=1}^{N^2}\sum_{m=1}^{2!} p_{n_1 m_1}^{ij} p_{n_2 m_2}^{ks},$$
$$E[q_{ij}q_{ks}q_{lr}] = \sum_{n=1}^{N^3}\sum_{m=1}^{3!} p_{n_1 m_1}^{ij} p_{n_2 m_2}^{ks} p_{n_3 m_3}^{lr},$$
$$E[q_{ij}q_{ks}q_{lr}q_{tu}] = \sum_{n=1}^{N^4}\sum_{m=1}^{4!} p_{n_1 m_1}^{ij} p_{n_2 m_2}^{ks} p_{n_3 m_3}^{lr} p_{n_4 m_4}^{tu} \quad (7)$$

where $[n_1\ n_2\ \ldots\ n_L]$ is the nth L-tuple of a set of N elements and $[m_1\ m_2\ \ldots\ m_L]$ is the mth permutation of the nth L-tuple. Efficient algorithms to obtain the tuples and the permutations can be found in [4]. The terms p_{nm}^{ij} of (7) are given by

$$p_{nm}^{11} = V_n^{11}(V_m^{11})^* + V_n^{21}(V_m^{21})^*, \quad p_{nm}^{12} = V_n^{12}(V_m^{11})^* + V_n^{22}(V_m^{21})^*,$$
$$p_{nm}^{21} = V_n^{11}(V_m^{12})^* + V_n^{21}(V_m^{22})^*, \quad p_{nm}^{22} = V_n^{12}(V_m^{12})^* + V_n^{22}(V_m^{22})^* \quad (8)$$

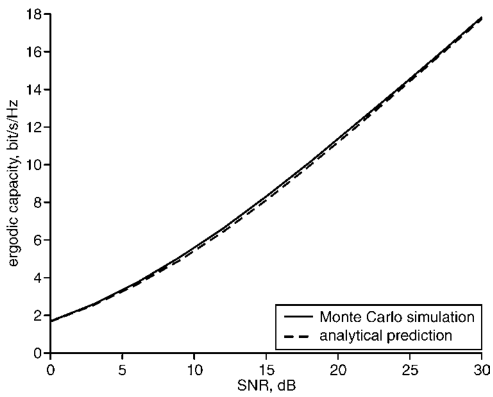

Fig. 1 *Comparison between theoretical ergodic capacity and analytical prediction of (4) for uncorrelated Rayleigh channel against SNR*

Simulation results: The expression (4) can be used to estimate the ergodic capacity in any channel defined by the multipath terms V_n^{ij}. For example, in Fig. 1 we compare the ergodic capacity of an uncorrelated Rayleigh MIMO channel (obtained by Monte Carlo simulation) with the analytical prediction of (4). The uncorrelated MIMO channel has been modelled as an $N = 20$ wave channel where the multipath terms V_n^{ij} have equal amplitude and independent random phases. It can be shown that this multipath channel approximately behaves as an uncorrelated Rayleigh channel [2]. The Figure shows that (4) approximates quite well the ergodic capacity for any SNR.

To validate (4) in a variety of propagation conditions (channels), we have simulated the ergodic capacity in a specific microcellular environment depicted in Fig. 2. It consists of 48 regularly distributed buildings with uniform height (25 m) and rectangular sections, forming a rectilinear grid of streets. This scenario represents an area of midtown Manhattan that has been traditionally used in the validation of a number of outdoor propagation models. The area of the environment is 900 × 500 m. In the simulations we assume that the Tx and Rx antennas are dual linear-polarised and omnidirectional for both polarisations. The Tx and Rx antennas are vertically oriented and located at 20 m and 1.5 m height, respectively. The simulations have been carried out at 1.8 GHz for different locations of the Rx antenna along line 1 in Fig. 2. In all the Rx locations the average SNR is assumed to be 15 dB. The multipath terms V_n^{ij} have been calculated using a 3D ray-tracing propagation tool, which takes into account multiple reflections from ground and buildings as well as diffractions around the wedges of the building walls and roofs [5]. Fig. 3 compares the analytical prediction with Monte Carlo simulations showing that (4) constitutes a tight approximation for the ergodic capacity in different propagation conditions.

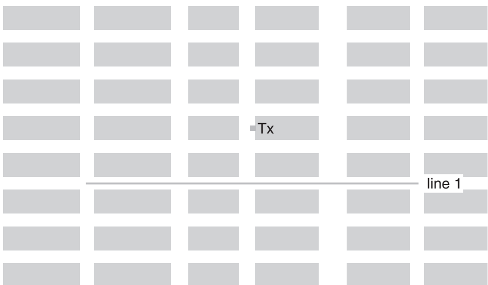

Fig. 2 *Top view of urban microcell showing Tx antenna location*
Line 1 shows Rx antenna locations in simulations

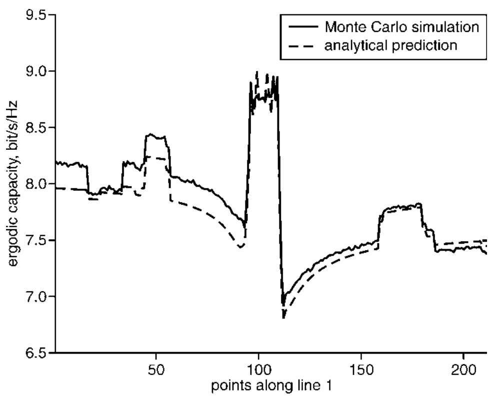

Fig. 3 *Simulated ergodic capacity along line 1 of Fig. 2 for SNR = 15 dB*
Figure compares the analytical prediction with Monte Carlo simulations

Acknowledgment: This work has been partially supported by the Spanish Ministry of Science and Technology under project TIC2001-0751-C04-03.

19 July 2004
Electronics Letters online no: 20046265
doi: 10.1049/el:20046265

J. Pérez, J. Ibáñez, L. Vielva and I. Santamaría (*Universidad de Cantabria, ETSIIT, Avda. Los Castros S/N., Santander 39005, Spain*)

E-mail: jperez@gtas.dicom.unican.es

References

1 Paulraj, A., Nabar, R., and Gore, D.: 'Introduction to space-time wireless communications' (Cambridge University Press, 2003)
2 Durgin, G.D.: 'Space-time wireless channels' (Prentice Hall PTR, Upper Saddle River, NJ, USA, 2003)
3 Foschini, G.J., and Gans, M.J.: 'On limits of wireless communications in a fading environment when using multiple antennas', *Wirel. Pers. Commun.*, 1998, **6**, (3), pp. 311–335
4 Knuth, D.E.: 'The art of computer programming IV' (Adison-Wesley, 1997–98)
5 Catedra, M.F., and Perez, J.: 'Cell planning for wireless communications' (Artech House, Norwood, MA, USA, 1999)