Closed-form Approximation for the Outage Capacity of Orthogonal STBC

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***Abstract*— In this letter we derive a tight analytical approximation for the outage capacity of orthogonal space-time block codes (STBC's). The proposed expression is a simple closed-form function of the power covariance matrix of the channel. In the case of uncorrelated channels, the expression only depends on the variances of the channel power gains that can be expressed analytically for the most common fading distributions: Rayleigh, Rice, Nakagami, Weibull, etc. Furthermore, the approximation encompasses different fading distributions and gains between different pairs of transmit and receive antennas, which can ocurr in distributed STBC networks.**

***Index Terms*— MIMO systems, channel capacity, space-time block codes (STBC's), fading channels.**

I. Introduction

It is well known that space-time block coding (STBC) transforms the multiple-input-multiple-output (MIMO) channel into a number of independent scalar channels providing diversity gain with very simple encoding and decoding [1], [2]. On the other hand, the achieved data rate is well below the theoretic capacity limit of the MIMO channel because the capacity of the effective scalar channel is lower than the capacity of the MIMO matrix channel [3]. In [4] a closed-form expression of information outage probability was derived for MIMO-OSTBC Nakagami-m fading channels [4]. In this letter we derive a general, simple and tight closed-form approximation for the outage capacity of MIMO-OSTBC systems that is valid for arbitrary channel fading distributions. The approximation also encompasses different fading distributions and gains between different pairs of transmit and receive antennas, which can be useful in the performance analysis of distributed STBC networks. The rest of the letter is organized as follows. In section II we derive the closed-form approximation for the outage capacity. In section III we show how to use the derived expression in different fading channels. Simulation results presented in section IV demostrate the accuracy of the approximation. Finally, some concluding remarks are given in section V.

II. Outage Capacity Approximation

We assume a frequency-flat fading channel, which is known at the receiver but unknown at the transmitter. We also assume i.i.d. AWGN noise at the receive antennas. When OSTBC is used, the MIMO channel transforms into an effective scalar complex AWGN channel with effective signal-to-noise-ratio (SNR) at detection given by [5],[3], [6], [7]

$$\gamma = \frac{E_s\, \alpha \left\|\mathbf{H}\right\|_F^2}{\sigma^2 n_T R} = \rho \frac{\left\|\mathbf{H}\right\|_F^2}{n_T R}, \tag{1}$$

where E_s is the total transmitted energy on the n_T transmit antennas per symbol time, α is the average path-loss between the transmitter and the receiver, R is the code rate, σ^2 is the noise power and $\left\|\mathbf{H}\right\|_F^2$ is the squared Frobenius norm of the $n_R \times n_T$ MIMO channel matrix $\mathbf{H}$. In (1) we assume that $\mathbf{H}$ is normalized so, $E\left[\left\|\mathbf{H}\right\|_F^2\right] = n_R n_T$, being $E\left[\cdot\right]$ the expectation operator. Notice that $\rho = E_s\,\alpha/\sigma^2$ is the average SNR at the receiver branches before decoding. The capacity (in bps/Hz) will be

$$C = R\log_2\left(1+\gamma\right). \tag{2}$$

Expanding (2) in Taylor series about the expected value of the effective SNR (μ_γ)

$$C\left(\gamma\right) = R\log_2\left(1+\mu_\gamma\right) + R\log_2\left(e\right)\sum_{m=1}^{\infty}\frac{\left(-1\right)^{m-1}}{m}\frac{\left(\gamma-\mu_\gamma\right)^m}{\left(1+\mu_\gamma\right)^m}, \tag{3}$$

where e is the Neper's number. Applying the expectation operator to (3), the second-order approximation for the ergodic capacity will be

$$E\left[C\right] = \mu_C \approx R\log_2\left(1+\mu_\gamma\right) - \frac{R\,\sigma_\gamma^2\,\log_2 e}{2\left(1+\mu_\gamma\right)^2}. \tag{4}$$

Similarly, expanding $C^2\left(\gamma\right)$ in Taylor series about μ_γ and applying the expectation operator, the second moment of the capacity can be approximated as follows

$$E\left[C^2\right] \approx R^2\left(\log_2\left(1+\mu_\gamma\right)\right)^2 + \frac{R^2\sigma_\gamma^2\,\log_2 e}{\left(1+\mu_\gamma\right)^2}\log_2\left(\frac{e}{1+\mu_\gamma}\right). \tag{5}$$

From (4) and (5), the variance of the capacity will be

$$\sigma_C^2 \approx R^2\left(\log_2 e\right)^2\left[\frac{\sigma_\gamma^2}{\left(1+\mu_\gamma\right)^2} - \frac{\sigma_\gamma^4}{4\,\left(1+\mu_\gamma\right)^4}\right]. \tag{6}$$

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Considering (1) and the channel normalization, the mean and variance of the capacity can be approximated as follows

$$\mu_C \approx R \log_2\left(1+\frac{\rho n_R}{R}\right) - \frac{R \log_2(e)\, \rho^2 \operatorname{var}\|\mathbf{H}\|_F^2}{2\, n_T^2 \left(R+\rho n_R\right)^2}, \quad (7)$$

$$\sigma_C^2 \approx \left(\frac{R\,\rho\, \log_2 e}{n_T\left(R+\rho n_R\right)}\right)^2 \operatorname{var}\|\mathbf{H}\|_F^2 \times\left(1-\frac{\rho^2 \operatorname{var}\|\mathbf{H}\|_F^2}{n_T^2\left(R+\rho n_R\right)^2}\right). \quad (8)$$

From (7) and (8) we can obtain a gaussian approximation of the cumulative distribution function of the capacity

$$F_C(c) \approx 1-\frac{1}{2} \operatorname{erfc}\left(\frac{c-\mu_C}{\sqrt{2}\sigma_C}\right), \quad (9)$$

where $\operatorname{erfc}(x)$ is the complementary error function. The $q\%$-outage capacity (C_q) is defined as the transmission rate that is guaranteed for $1-q/100$ of the channel realizations. Then, from (9), the q% outage capacity can be approximated as follows

$$C_q \approx \mu_C+\sigma_C \sqrt{2} \operatorname{erfc}^{-1}\left(2-\frac{q}{50}\right). \quad (10)$$

The expressions (7), (8) and (10) reveal that when the number of antennas increases, the capacity variance decreases and the ergodic and outage capacity become less dependent on the channel fading statistics and on the number of transmit antennas.

III. Fading Channel Distributions

Equation (10), combined with (7) and (8), provides us with a closed-form approximation of the outage capacity as a function of the variance of $\|\mathbf{H}\|_F^2$. This can be expressed as follows

$$\operatorname{var}\|\mathbf{H}\|_F^2 = \sum_{i=1}^{n_R} \sum_{j=1}^{n_T} \sum_{k=1}^{n_R} \sum_{s=1}^{n_T} E\left[\left|h_{ij}\right|^2\left|h_{ks}\right|^2\right] - E\left[\left|h_{ij}\right|^2\right] E\left[\left|h_{ks}\right|^2\right], \quad (11)$$

where the h_{ij} are the entries of the MIMO channel matrix $\mathbf{H}$. Notice that the variance of $\|\mathbf{H}\|_F^2$ is the sum of the entries of the power covariance matrix of the channel [8], [9]. Therefore, the power covariance matrix is the only channel statistic required for the capacity estimation. Much of the experimental research in antenna diversity has involved the measurements of power covariance matrices. Assuming uncorrelated channels (11) reduces to

$$\operatorname{var}\|\mathbf{H}\|_F^2 = \sum_{i=1}^{n_R} \sum_{j=1}^{n_T} \operatorname{var}\left|h_{ij}\right|^2. \quad (12)$$

Therefore, for uncorrelated channels we only need the variances of the power gains to estimate the outage capacity. These variances can be obtained analytically for most of the fading distributions. Assuming that the Ω_{ij} are the mean powers of the channel matrix entries (they determine the potential imbalances of the channel)

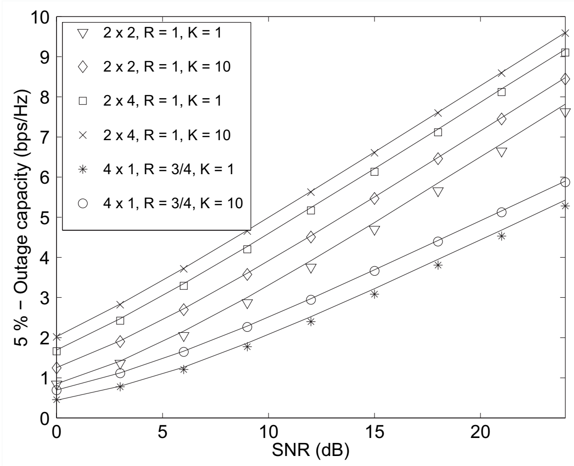

Fig. 1. 5%-outage capacity as a function of the average SNR at the receiver (before decoding) for some uncorrelated MIMO ricean fading channels with different number of antennas, code rates (R) and ricean-K factors (K).

$$\text{Nakagami-m:} \quad \operatorname{var}\left|h_{ij}\right|^2 = \frac{\Omega_{ij}^2}{m_{ij}}, \quad (13)$$

$$\text{Rice:} \quad \operatorname{var}\left|h_{ij}\right|^2 = \Omega_{ij}^2 \times\left(\frac{2 \exp\left(-K_{ij}\right)\ {}_1F_1\left(3;1;K_{ij}\right)}{\left(K_{ij}+1\right)^2}-1\right), \quad (14)$$

$$\text{Weibull:} \quad \operatorname{var}\left|h_{ij}\right|^2 = \Omega_{ij}^2\left(\frac{\Gamma\left(1+4/\beta_{ij}\right)}{\left(\Gamma\left(1+2/\beta_{ij}\right)\right)^2}-1\right), \quad (15)$$

where the m_{ij} are the Nakagami parameters, the K_{ij} are the ricean factors and the β_{ij} are the Weibull parameters. ${}_1F_1(\cdot;\cdot;\cdot)$ is the confluent hypergeometric function and $\Gamma(\cdot)$ is the gamma function.

Similarly, other fading distributions can be considered. Notice also that the Rayleigh distribution can be viewed as a particular case of the Nakagami distribution with $m_{ij}=1$, Ricean distribution with $K_{ij}=0$ or Weibull distribution with $\beta_{ij}=2$.

IV. Simulation Results

To show the accuracy of the derived approximation we compare its predictions with those obtained through Monte Carlo simulations in a variety of MIMO configurations and fading distributions. In all the cases, the solid lines represents the predictions provided by (10) and the markers are the values obtained by Monte Carlo simulations.

Fig.1 shows the 5%-outage capacity versus the average SNR at the receiver before decoding (ρ in (1)) for uncorrelated identically distributed ricean fading channels. MIMO ricean channels with different number of antennas, code rates (R) and ricean-K factors are considered. Fig.2 shows the outage capacity versus outage probability (q) for uncorrelated 4×4 weibull fading channels with different code rates and Weibull

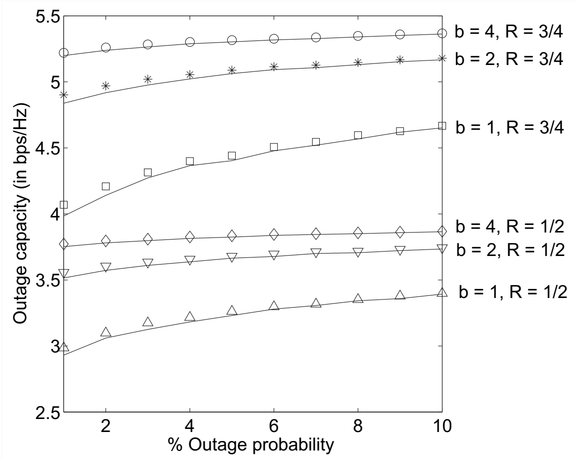

Fig. 2. Outage capacity versus outage probability for 4×4 uncorrelated Weibull fading channels with different code rates (R) and Weibull fading parameters (b). The SNR before decoding is $15\,dB$ in all the cases.

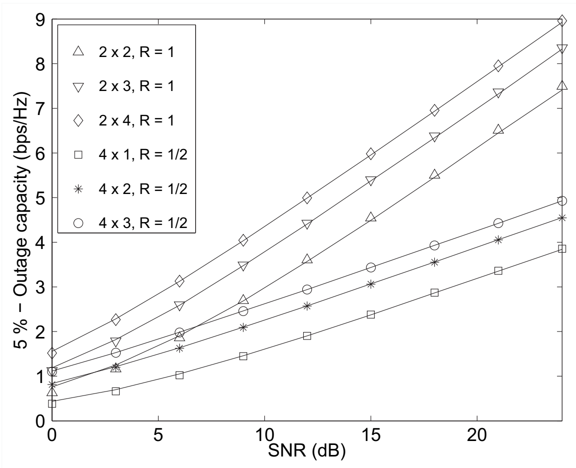

Fig. 3. 5%-outage capacity versus the average SNR at the receiver for different uncorrelated Nakagami fading channels where the channel matrix entries are not identically distributed.

fading parameters (b). In this case the average SNR before decoding is $15\,dB$.

In the previous results we have considered channel matrices with identically distributed entries. In distributed networks the fading distributions between different pairs of transmit and receive antennas can be different. In these cases the proposed expression can also be used providing accurate approximations. As an example Fig. 3 shows results for different Nakagami-m fading channels where the channel matrix entries are not identically distributed. In the configurations with two transmit antennas the Nakagami parameters were $m_{i1} = 0.5,\ m_{i2} = 2,\ i = 1, \ldots, n_R$, and in the cases with four transmit antennas the Nakagami parameters were $m_{i1} = 0.5,\ m_{i2} = 1,\ m_{i3} = 2,\ m_{i4} = 4,\ i = 1, \ldots, n_R$.

V. Conclusions

In this letter we have derived a tight and simple closed-form approximation for the outage capacity of narrow-band MIMO-OSTBC systems that is valid for any fading channel. The expression is a simple function of the power covariance matrix of the channel. In uncorrelated channels, the approximation only depends on the variances of the power gains of the MIMO channel. Such variances can be analytically expressed for the most common fading distributions: Rayleigh, Rice, Nakagami, Weibull, etc. The proposed expression is also useful for MIMO channels with different fading distributions, which is the case of many distributed MIMO-STBC networks. The accuracy of the approximation reveals that, for uncorrelated channels, the ergodic and outage capacities mainly depend on the means and variances of the power gains of the MIMO channel regardless the specific fading distributions. The derived expression also shows that, for high number of antennas, the ergodic and outage capacities are little dependent on the channel fading statistics and on the number of transmit antennas.

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