A general test to check the feasibility of linear interference alignment

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***Abstract*—In this paper, we propose a test for checking the feasibility of linear interference alignment (IA) for multiple-input multiple-output (MIMO) channels with constant coefficients for any number of users, antennas and streams per user. We consider the compact complex manifold formed by those channels, precoders and decoders that satisfy the polynomial IA equations (the so-called solution variety), and study its projection onto the input space formed by the interference channels. When the derivative of this projection is surjective, namely when the tangent space of the solution variety is projected into the whole tangent space of the inputs space, the linear alignment problem is feasible; otherwise is infeasible. Building on these results, a general feasibility test, which amounts to check whether a given matrix is full-rank or not, is proposed.**

I. Introduction

It has been recently shown that to achieve the maximum spatial degrees of freedom (DoF) of the K-user multiple-input multiple-output (MIMO) interference channel, the interference from other transmitters must be aligned at each receiver in a lower-dimensional subspace. This is the basic idea of the interference alignment (IA) technique, which was first proposed in [1], [2] and has received a lot of attention since then.

In this paper, we restrict our attention to alignment schemes in signal vector space (i.e., schemes that apply linear precoding and decoding), and address the feasibility of linear IA for MIMO interference networks with constant channel coefficients, which remains as an open problem in its full generality. The first work to study this problem was [3], where the solvability of the IA polynomial equations was analyzed based on classic results in algebraic geometry like Bezout's and Bernstein's theorems. By counting the number of equations and variables involved in any subset of zero-forcing alignment equations, Yetis *et al*. introduced in [3] the definition of a *proper* system. Connections between *proper* and feasible systems were established only for the single-beam case in which each user transmits only one stream of data, whereas for the multibeam case the equivalence between *proper* and feasible systems does not longer hold.

In [4] the feasibility of IA was studied by interpreting the alignment process as a joint transmit-receive zero-forcing scheme in which each interfering stream can be suppressed at either the transmitter or the receiver sacrificing one degree of freedom. The proposed feasibility test, however, provides only necessary conditions and is combinatorial in nature since must check all possible ways to suppress interfering streams at both sides of the link and for all users.

More recent work on the feasibility of IA has been presented in [5] and [6]. Specifically, [5] studies the dimensions of the algebraic varieties involved in the alignment problem (input, output and solution variety), and proves a sufficient and necessary condition of feasibility for the particular case of symmetric square MIMO interference channels, where all transmitters and receivers have the same number of antennas, all users transmit the same number of streams and there are at least three interfering users ($K \geq 3$). For the general case with arbitrary system parameters, only a necessary condition is proved in [5]. Similar algebraic tools are used in [6] to prove general bounds on the tuple of DoF that are achievable through linear interference alignment. Furthermore, for the particular case of symmetric systems where the number of transmit and receive antennas is divisible by the number of streams the bound is tight and can be achieved through IA.

In this paper we propose a polynomial-time complexity test of feasibility for the linear alignment problem. The main idea is to consider the input space (projective space of MIMO interference channels) and the solution variety (channels, precoders and decoders satisfying the IA polynomial equations) as smooth compact manifolds and to study the linear mapping between the tangent spaces of both smooth manifolds given by the first projection. It the mapping is surjective, which amounts to check whether a given matrix is full-rank, the linear alignment is feasible; otherwise is infeasible. We show that the test is consistent with all known DoF outer bounds for the interference channel (e.g., [7], [8], [9], [10], [11]). Furthermore, the proposed feasibility test can be used to obtain the total DoF for any K-user MIMO interference channel simply by checking the feasibility of all possible DoF tuples.

II. Problem Statement

We consider in this paper the K-user MIMO interference channel with transmitter j having $M_j \geq 1$ antennas and receiver j having $N_j \geq 1$ antennas. We assume without loss of generality that each user wishes to send $d_j \geq$

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1 streams or messages[1]. We adhere to the notation used in [3] and denote this asymmetric interference channel as $\prod_{k=1}^{K}(M_k \times N_k, d_k) = (M_1 \times N_1, d_1)\cdots(M_K \times N_K, d_K)$. The symmetric case in which all users transmit d streams and are equipped with M transmit and N receive antennas is denoted as $(M \times N, d)^K$. In the square symmetric case all users have the same number of antennas $M = N$.

The MIMO channel from transmitter l to receiver k is denoted as H_{kl} and assumed to be flat-fading and constant over time. Each H_{kl} is an $N_k \times M_l$ complex matrix with independent entries drawn from a continuous distribution (channels generated in this way are generic). For the sake of simplicity, we will consider only the fully connected interference channel where each user receives interference from the rest of $K-1$ transmitters, and where each transmitter interferes to all the $K-1$ unintended receivers. However, the results in this paper can be easily extended to partially connected networks.

User j encodes its message using an $M_j \times d_j$ precoding matrix V_j and the received signal is given by

$$y_j = H_{jj}V_j x_j + \sum_{i\neq j} H_{ji}V_i x_i + n_j, \qquad (1)$$

where x_j is the $d_j \times 1$ transmitted signal and n_j is the zero mean unit variance circularly symmetric additive white Gaussian noise vector. The first term in (1) is the desired signal, while the second term represents the interference space. The receiver j applies a linear decoder U_j of dimensions $N_j \times d_j$, i.e.,

$$U_j^T y_j = U_j^T H_{jj}V_j x_j + \sum_{i\neq j} U_j^T H_{ji}V_i x_i + U_j^T n_j, \qquad (2)$$

where superscript T denotes transpose.

The interference alignment (IA) problem consist in finding the precoders and decoders, V_j and U_j, in such a way that the interfering signals at each receiver fall into a reduced-dimensional subspace and the receivers can then extract the projection of the desired signal that lies in the interference-free subspace. To this end, it is required that the polynomial equations

$$U_k^T H_{kl} V_l = 0, \qquad k \neq l, \qquad (3)$$

are satisfied, while the signal subspace for each user must be linearly independent of the interference subspace and must have dimension d_k, that is

$$\operatorname{rank}(U_k^T H_{kk} V_k) = d_k, \qquad \forall\, k \in \{1,\ldots,K\}. \qquad (4)$$

Taking (4) into account, we assume in the sequel that for any feasible IA strategy the number of streams transmitted by each user satisfies the point-to-point bound given by

$$1 \leq d_k \leq \min\{N_k, M_k\}, \qquad \forall\, k \in \{1,\ldots,K\}. \qquad (5)$$

[1]If a user has $d_k = 0$, it can be removed from the interference channel.

A. Feasibility of IA

In this paper we are interested in studying the following feasibility problem: For a given interference network, $\prod_{k=1}^{K}(N_k \times M_k, d_k)$, to decide whether there exists a linear alignment solution for *generic* interference MIMO channels H_{kl}. In that case, we say that the problem is generically feasible, or simply feasible.

To formally state the feasibility problem, let us consider the following diagram

$$\begin{array}{ccc} & \mathcal{V} & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ \mathcal{H} & & \mathcal{S} \end{array} \qquad (6)$$

where

$$\mathcal{H} = \prod_{k\neq l} \mathbb{P}(\mathcal{M}_{N_k \times M_l}(\mathbb{C})) \qquad (7)$$

is the input space composed by the Cartesian product of all interference MIMO channels (here, $\mathbb{P}(\mathcal{M}_{N_k \times M_l}(\mathbb{C})$ denotes the projective space of complex-valued $N_k \times M_l$ matrices),

$$\mathcal{S} = \mathbb{G}_{d_1,N_1} \times \cdots \times \mathbb{G}_{d_K,N_K} \times \mathbb{G}_{d_1,M_1} \times \cdots \times \mathbb{G}_{d_K,M_K}$$

is the output space formed by the precoder and decoder Grassmannians ($\mathbb{G}_{a,b}$ denotes the Grassmannian formed by the linear subspaces of complex dimension a in $\mathbb{C}^b$, with $0 \leq a \leq b$); and

$$\mathcal{V} = \{(H,U,V) \in \mathcal{H} \times \mathcal{S} : \text{ (3) holds}\}$$

is the so–called solution variety. Note that we are denoting by H the collection of all matrices H_{kl}, and, similarly, U and V denote the set of U_j and V_j, $1 \leq j \leq K$, respectively. Defined in this way, $\mathcal{H}$, $\mathcal{S}$ and $\mathcal{V}$ are all compact and smooth manifolds [13]. Specifically, the set $\mathcal{V}$ is given by certain polynomial equations, linear in each of the H, U, V and therefore is an algebraic subvariety of the product space $\mathcal{H} \times \mathcal{S}$.

Note that, given $H \in \mathcal{H}$, the set $\pi_1^{-1}(H)$ is a copy of the set of U, V such that (3) holds, that is the solution set of the linear interference alignment problem. On the other hand, given $(U, V) \in \mathcal{S}$, the set $\pi_2^{-1}(U, V)$ is a copy of the set of $H \in \mathcal{H}$ such that (3) holds. The feasibility question studied in this paper can then be restated more formally as follows,

Problem 1: Is $\pi_1^{-1}(H) \neq \emptyset$ for a generic H?

III. Main Result

The following theorem solves the question posed in Problem 1 and, as a by product, provides a simple feasibility test.

Theorem 1: Problem 1 is feasible iff for almost every choice of H_{kl}, and for any choice of U_k, V_l satisfying (3), the linear mapping, θ, defined by

$$(\dot{U}_1,\ldots,\dot{U}_K,\dot{V}_1,\ldots,\dot{V}_K) \mapsto \left\{\dot{U}_k^T H_{kl} V_l + U_k^T H_{kl} \dot{V}_l\right\} \qquad (8)$$

is surjective, i.e., it has maximal rank equal to $\sum_{k\neq l} d_k d_l$. Here, $\left(\dot{U}_1,\ldots,\dot{U}_K,\dot{V}_1,\ldots,\dot{V}_K\right)$ denotes a set of matrices of dimensions $N_k \times d_k$ or $M_l \times d_l$ depending on whether $\dot{U}_k$ or $\dot{V}_l$ is considered, which are affine representations of the components of a vector in the tangent space of $\mathcal{V}$.

A. Proof sketch and mathematical insights

A detailed proof of the claim in Theorem 1 can be found in [13]. Here, due to space limitations, we only provide a sketch of the proof along with a more descriptive explanation that help readers to develop a clear understanding of our main result. It is not difficult to see that $\mathcal{V}$ is not only an algebraic set but also a compact complex manifold, whose dimension can be easily computed from the fact that its constituent equations are, when regarded as affine equations, independent. One can also (trivially) compute the dimension of $\mathcal{H}$ and $\mathcal{S}$. If $\dim \mathcal{V} < \dim \mathcal{H}$, then the projection $\pi_1(\mathcal{V})$ cannot cover most of $\mathcal{H}$. It turns out that this is indeed the case that the problem is not *proper* as defined in [3]. Let us center our attention in the *proper* case: we have $\dim \mathcal{V} \geq \dim \mathcal{H}$, but it can still happen that $\pi_1(\mathcal{V})$ is only a zero–measure set of $\mathcal{H}$. The reader can have in mind a vertical line projecting into a horizontal line: both sets have the same dimension but the projection of the vertical line is just a point, thus a zero measure subset of the horizontal line. Mathematically, this is identified by the fact that for every element of $\mathcal{V}$, the derivative of π_1 is not surjective, namely it does not project the tangent space of $\mathcal{V}$ onto the whole tangent space of $\mathcal{H}$, just on a vector subspace. Now, the tangent space of $\mathcal{V}$ is easy to describe: it is the set of tangent vectors $(\dot{H}, \dot{U}, \dot{V})$ such that

$$\dot{U}_k^T H_{kl} V_l + U_k^T \dot{H}_{kl} V_l + U_k^T H_{kl} \dot{V}_l = 0, \quad k \neq l.$$

Because U_k and V_l are full–rank, the fact that the derivative of π_1 is surjective is then equivalent to the mapping (8) being surjective (this is not a completely obvious fact, see [13] for a proof.) Thus, checking the rank of θ detects if the projection from $\mathcal{V}$ onto $\mathcal{H}$ has a surjective derivative at some chosen point. Now, standard tools from differential topology[2] and algebraic geometry prove that:

- either θ is almost everywhere surjective, in which case the problem is feasible or
- θ is nowhere surjective, in which case the problem is infeasible.

This is the main idea behind the proof of our Theorem 1. Note that choosing at random some $(H, U, V) \in \mathcal{V}$ and checking if the associated linear mapping (8) is surjective permits to decide if the IA problem is feasible or not. If the test answers "feasible" then the problem is feasible. If it answers "infeasible" then, unless we were extremely unlucky and chose (H, U, V) in a certain zero–measure set, the problem is infeasible (and thus our answer is correct). In practice, one can repeat the test a few times to discard that possibility.

IV. Feasibility Test

As indicated by Theorem 1, the proposed feasibility test has to perform two tasks:

1) To choose an arbitrary set (H_{kl}, U_k, V_l) such that (3) holds: we call this the *inverse* IA problem.
2) To check whether the matrix B (in any basis) defining the linear mapping (8) satisfies $\det(BB^*) \neq 0$, which means that the mapping θ is surjective.

Now, we detail both stages of the proposed IA feasibility test.

A. Solving the inverse IA problem

The inverse IA problem consists of, given arbitrary (random) realizations of U_k, V_l, finding a set of MIMO channels such that (3) holds. Since this problem is linear in the entries of the MIMO channels, it amounts to solving the following set of linear systems of equations

$$A_{kl} \operatorname{vec}(H_{kl}) = 0, \quad k \neq l, \tag{9}$$

where $\operatorname{vec}(H_{kl})$ is a vectorized version of the MIMO channel matrix H_{kl}, and A_{kl} is an $d_k d_l \times N_k M_l$ matrix obtained as the transpose of the Kronecker product of V_l and U_k, that is

$$A_{kl} = (V_l \otimes U_k)^T, \quad k \neq l. \tag{10}$$

For example, for the 3-user interference channel, the inverse IA problem reduces to solving the following linear systems of equations

$$(V_l \otimes U_k)^T \operatorname{vec}(H_{kl}) = 0, \tag{11}$$

where $(k, l) \in \{(1,2), (1,3), (2,1), (2,3), (3,1), (3,2)\}$.

B. Checking the rank of the linear mapping θ

The solution of the inverse IA problem obtained in the previous stage, provide us with a generic choice of U_k, V_l and H_{kl} satisfying (3). For this particular element of the solution variety, the linear mapping θ given by (8) can be written in matrix form as

$$Bw, \tag{12}$$

where $w = \operatorname{vec}([\dot{U}_1, \ldots, \dot{U}_K, \dot{V}_1, \ldots, \dot{V}_K])$ and B is a block matrix with $K(K-1)$ row partitions (as many blocks as interfering links) and $2K$ column partitions (as many blocks as precoding and decoding matrices). Checking the feasibility of IA then reduces to check whether the matrix B is full rank or not. The vectorization of the mapping (8) shows that B is composed of two main kinds of blocks, $B_{kl}^{(U)}$ and $B_{kl}^{(V)}$, i.e.

$$\begin{aligned} &\operatorname{vec}(\dot{U}_k^T H_{kl} V_l + U_k^T H_{kl} \dot{V}_l) = \\ &\underbrace{(I_{d_l} \otimes U_k^T H_{kl})}_{B_{kl}^{(U)}} \operatorname{vec}(\dot{V}_l) + \underbrace{(V_l^T H_{kl}^T \otimes I_{d_k}) K_{N_k, d_k}}_{B_{kl}^{(V)}} \operatorname{vec}(\dot{U}_k), \end{aligned} \tag{13}$$

where I_n denotes the $n \times n$ identity matrix and $K_{m,n}$ is the $mn \times mn$ commutation matrix which is defined as the matrix that transforms the vectorized form of an $m \times n$ matrix into the vectorized form of its transpose. Block $B_{kl}^{(U)}$ has dimensions $d_l d_k \times d_l M_l$, whereas block $B_{kl}^{(V)}$ is $d_l d_k \times d_k N_k$. For a given tuple (k, l), $B_{kl}^{(U)}$ and $B_{kl}^{(V)}$ are placed in the row partition that corresponds to the interfering link indicated by the tuple (k, l). However, $B_{kl}^{(U)}$ is placed in the $l + K$-th column partition, whereas $B_{kl}^{(V)}$ occupies the k-th column partition. The rest of blocks are occupied by null matrices

[2]Remarkably, a theorem by Ehresmann [12] that strongly uses the fact that $\mathcal{V}$ is compact and that the base field is $\mathbb{C}$ and not $\mathbb{R}$.

with all entries being zero. The dimensions of B are therefore $\sum_{k\neq l} d_k d_l \times \sum_{j=1}^{K}(M_j+N_j)d_j$.

Once B has been built, the last step is to check if its rank is maximum, that is, $\mathrm{rank}(B)=\sum_{k\neq l} d_k d_l$. A simple method consists of generating a random element b in $\mathbb{C}^{\sum_{k\neq l} d_k d_l}$, computing the least squares solution of $Bw=b$ and checking if $\|Bw-b\|$ is small enough, say smaller than or equal to 10^{-8}. With high probability in the choice of b this test will determine if θ is a surjective mapping.

Notice that B has the same structure as the incidence matrix of the network connectivity graph. Taking again the 3-user interference channel as an example, B is constructed as follows

$$\begin{bmatrix} B_{12}^{(V)} & 0 & 0 & 0 & B_{12}^{(U)} & 0 \\ B_{13}^{(V)} & 0 & 0 & 0 & 0 & B_{13}^{(U)} \\ 0 & B_{21}^{(V)} & 0 & B_{21}^{(U)} & 0 & 0 \\ 0 & B_{23}^{(V)} & 0 & 0 & 0 & B_{23}^{(U)} \\ 0 & 0 & B_{31}^{(V)} & B_{31}^{(U)} & 0 & 0 \\ 0 & 0 & B_{32}^{(V)} & 0 & B_{32}^{(U)} & 0 \end{bmatrix}$$

where the blocks $B_{kl}^{(U)}$ and $B_{kl}^{(V)}$ are given by (13).

Remark 1: The above test has been presented in floating point arithmetic and is thus sensitive to floating point errors. Although it is robust enough for every example that we have tried, it is straightforward to design a version which works in exact arithmetic, as shown in [13]. Furthermore, regarding the complexity of the proposed test, it can be shown that the problem of deciding if a given system is generically infeasible is in the BPP (Bounded-error Probabilistic Polynomial time) complexity class.

Remark 2: Some complexity analysis have recently appeared in the literature suggesting that to check the feasibility of IA problems is NP-hard [14]. However, there is a crucial difference between the problem considered in [14] and the one considered in this paper. Informally stated, the key difference is that [14] studies the alignment problem for *a particular realization* of the interference channels, H_{kl}, whereas we consider the feasibility of IA for generic channels (e.g., channels with i.i.d. entries drawn from any continuous distribution). Notice also that even if the generic feasibility problem can be solved in polynomial time, finding the actual precoders and decoders that align the interference subspaces for a particular channel realization can still be NP-hard, as proved in [14], [15].

Remark 3: If the problem is feasible for a given stream distribution, $(d_1,\ldots,d_K)$, this DoF tuple is achievable by linear IA. Therefore, by solving the feasibility problem it would be possible to obtain the maximum total DoF for any interference channel simply by exhaustive search over all possible DoF tuples that satisfy the existing outer bounds.

V. Discussion and Examples

A. Congruence with existing feasibility results and DoF bounds

The feasibility problem is first considered in [3], where a system is classified as *proper* if and only if the number of independent variables in every set of equations in (3) is at least as large as the number of equations in that set. This result assumes the polynomial equations involved in the problem are generic. However, for multibeam scenarios this assumption does not hold, and a correspondence between *proper* and feasible systems cannot be established. Conversely, *improper* systems are infeasible, as proved in [5] and [6]. Our test is congruent with these results since *improperness* implies $\sum_{j=1}^{K}(M_j+N_j)d_j < \sum_{k\neq l} d_k d_l$ and, hence, $\mathrm{rank}(B) < \sum_{k\neq l} d_k d_l$.

Additionally, for the K-user symmetric channel, denoted as $(M\times N,d)^K$, *proper* systems are feasible if $M=N$ [5] or both M and N are divisible by d [6]. Recent results in [9], [10] and [11] give necessary and sufficient conditions for the feasibility of the 3-user symmetric scenario, $(M\times N,d)^3$. For all these particular settings, the proposed test is in agreement with known results.

In those scenarios where none of the previous feasibility results apply, we have to rely on DoF outer bounds for establishing the infeasibility of a *proper* system. For instance, it is well known that for a point-to-point MIMO channel, $(M_k\times N_k,d_k)$, (5) has to be verified. In addition, for any 2-user pair in the network, $(M_k\times N_k,d_k)(M_l\times N_l,d_l)$, the following outer bound must be satisfied [7],

$$d_k+d_l \le \min\left\{M_l+M_k, N_l+N_k, \max\{N_k,M_l\}, \max\{N_l,M_k\}\right\}. \tag{14}$$

For the symmetric K-user interference channel (with uneven stream distribution among the users), $\prod_{k=1}^{K}(M\times N,d_k)$, the following outer bound, proved in [8], can be applied

$$\sum_k d_k \le K\min(M,N)\, I(K\le R) + K\frac{\max(M,N)}{R+1} I(K>R), \tag{15}$$

where $R=\left\lfloor \frac{\max(M,N)}{\min(M,N)} \right\rfloor$ and $I(\cdot)$ represents the indicator function.

The proposed test is also consistent with all these outer bounds. In fact, the test is proven to answer *infeasible* when (5) is not satisfied, cf. [13]. Proving that the violation of other, more complicated, bounds like (14) and (15) also results in matrix B being rank-deficient will be considered in a future work.

B. Some representative examples

Now, let us discuss the results provided by our feasibility test in some representative examples[3].

Example 1: First, consider the symmetric $(5\times 11,4)^3$ interference network, which is classified as *proper* according to [3]. According to [9] and [11] this scenario is infeasible which is also in agreement with our test. Thus, the maximum total number of DoF for this network cannot be larger than 11 as implied also by the outer bound in (15). By shutting off one beam of the first user, the system $(5\times 11,3)(5\times 11,4)^2$ could, in principle, be feasible because it satisfies the mentioned outer

[3]The reader is invited to test the feasibility of an arbitrary alignment problem at http://www.gtas.dicom.unican.es/IAtest.

bound. Our test shows that this system is actually feasible and, thus, the outer bound (15) is tight for this particular scenario.

Example 2: Now, consider the 3-user system $\prod_{j=1}^{3}(7 \times 13, d_j)$ where the stream distribution among users is not specified. The results in [9], [11] show that the system is infeasible if 5 streams per user are transmitted. Our test indicates that the $(7 \times 13, 5)^3$ system is infeasible whereas the system $(7 \times 13, 4)(7 \times 13, 5)^2$ is feasible, which allows us to claim that the maximum total DoF for this network is 14.

Example 3: Consider the scenarios $(15 \times 5, 4)^4$ and $(24 \times 6, 5)^5$ which are the simplest non-trivial scenarios for 4 and 5 users, respectively, that are *proper* but infeasible. Both scenarios violate the outer bound (15) which establishes that the total DoF cannot exceed 15 and 24, respectively. In fact, the proposed test shows that $(15 \times 5, 3)(15 \times 5, 4)^3$ and $(24 \times 6, 4)(24 \times 6, 5)^4$ are feasible systems and, then, (15) is tight for these particular scenarios.

Example 4: The $(4\times4, 2)(5\times3, 2)(6\times2, 2)$ network, which was studied in [4], satisfies the 2-user outer bound (14) for all 2-user pairs in the network. No other existing results apply to asymmetric systems like this one. However, the test proposed in this paper answers *infeasible*.

Example 5: A final interesting example is the $(2\times2, 1)(5\times5, 2)^2(8 \times 8, 4)$ system, which is feasible according to the proposed test. This system has been built by taking the symmetric $(5 \times 5, 2)^4$ system, which is known to be feasible, and transferring 6 antennas from the first to the fourth user. It must be noticed that while the total amount of antennas for both systems remains constant, the redistribution of antennas has allowed to achieve one DoF more in the asymmetric network. This example gives new evidence for the conjecture settled in [4], which asserts that for a given total number of DoF, $d_{tot} = \sum_k d_k$, there exist feasible asymmetric MIMO interference systems (that is, with unequal antenna and stream distribution among the links) such that the total number of antennas, $\sum_k (M_k + N_k)$, is less than number of antennas of the smallest symmetric system ($M_k = M$, $N_k = N$, and $d_k = d_{tot}/K$) that can achieve d_{tot}.

Let us finally point out that, in all cases in which our feasibility test was positive, we were able to find an IA using the iterative interference leakage minimization algorithm proposed in [16][4].

VI. Conclusions

This paper gives a test to check the feasibility of linear interference alignment in arbitrary K-user MIMO interference channels. The test is a direct consequence of a powerful result which states that the linear alignment problem is generically feasible iff the algebraic dimension of the solution variety is larger than or equal to the dimension of the input space *and* the linear mapping between the tangent spaces of both smooth manifolds given by the first projection is surjective. In practice, our test reduces to check whether a given matrix is full rank or not. The test neither requires any assumption (apart from the genericity of the MIMO channels), nor relies on any information-theoretic bound for the interference channel. However, it always provides answers which are in agreement with all the existing outer bounds and the most recent feasibility results.

[4]Notice, however, that alternating minimization algorithms cannot guarantee convergence to a global minimum, so they cannot be used as feasibility tests.

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