

A REGULARIZED DIGITAL FILTERING TECHNIQUE FOR THE SIMULTANEOUS RECONSTRUCTION OF A FUNCTION AND ITS DERIVATIVES

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ABSTRACT

In this paper a new regularized digital filtering technique for the simultaneous approximation of a function and its derivatives is proposed. First, a simple and local method is presented that interpolates the specified sample values exactly. The solution obtained by this method belongs to the space of splines functions, and can be implemented using filter banks. Unfortunately, like most of the methods used to solve interpolation problems using derivatives, it is very sensitive to noise. To overcome this drawback we extend the interpolation method to function approximation by defining a regularized functional, which includes a term forcing the smoothness of the solution. The minimization of this functional is performed by solving a simple linear system of equations or using gradient descent based techniques. Some examples show the improved performance of this technique in noisy environments.

1. INTRODUCTION

In some applications it is necessary to get a close approximation of both a function and its derivatives. For instance, to predict the intermodulation behavior of a microwave transistor, it is necessary to approximate not only its current/voltage (I/V) nonlinear characteristic but also up to its third order derivative [1]. Another example is the work carried out by Jordan in robotics [2].

In such applications, we usually collect samples of a function and its derivatives and the goal is to use all this information in the reconstruction process. Moreover, there are some applications such as aircraft communications, traffic control simulation or telemetry where the derivatives are easily accessible, and to use them in the reconstruction of the function can be interesting even when there is not an explicit interest in the derivatives. In such cases, the motivation to do this would be to be able to sample less frequently or to obtain some degree of immunity in noisy environments.

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Several approaches have been proposed in the literature to solve this problem. For example, Neural Networks have the theoretical capability of approximating a function and its derivatives [3], although there are not constructive approaches in this direction. On the other hand, when the signal is known to be bandlimited we can use an extension of the Shannon sampling theorem [4], or the iterative method proposed by Razafinjato for irregular sampling [5]. Recently, a method using perfect reconstruction (PR) filter banks has been proposed [6]. Unlike the methods in [4, 5], the reconstructed function obtained by [6] belongs to the space of splines functions and therefore it is not bandlimited. All these approaches, however, present the drawback of a high sensitivity to noise.

To overcome this shortcoming, in this paper we present a new regularized reconstruction method in the space of polynomial splines. By relaxing the interpolation conditions we can impose additional constraints forcing the smoothness of the function. This can be helpful in noisy environments where a smooth solution can reduce the degradation in the reconstruction process. The proposed method is not restricted to the reconstruction of bandlimited signals, and it is applicable to both regular and irregular sampling.

2. INTERPOLATION IN THE SPACE OF SPLINES

Our problem can be stated as follows: given a set of samples of a function and its first D derivatives ($d = 0, \dots, D$):

$$x^{(d)}[n] = x^{(d)}(nT), \quad n = 0, \dots, N-1; \quad (1)$$

to reconstruct a function $x(t)$, which fulfills the interpolation conditions (1). For the sake of simplicity we will consider here the interpolation problem using only samples of the function and its first derivative ($D = 1$). The method, however, can be easily extended to higher derivatives and even to irregular sampling. Moreover, in the sequel the superindex $d = 0$ will be understood.

Our goal is to obtain a function in the space of splines of order $D + 1$ (i.e., quadratic splines for $D = 1$) that fulfills the interpolation conditions (1). This quadratic spline reconstruction can be performed through a set of synthesis functions obtained using perfect reconstruction filter bank theory as it is shown by Djokovic and Vaidyanathan in [6]. Here we describe an alternative method, which will be useful in the next section for the extension of the interpolation method to a regularized approximation.

The main idea of the method is that the original sequences of the function and its derivative $x^{(d)}[n]$, ($d = 0, 1$), can be interpolated by 2 to obtain the sequences $x_i^{(d)}[n]$, $n = 0, 1, \dots, 2N - 2$. Then, since the derivative of the function in the space of quadratic splines is piecewise-linear, we can use this knowledge and the constraints imposed by the continuity of the function and its derivative to obtain the samples of the interpolated sequences $x_i^{(d)}[n]$, $n = 0, 1, \dots, 2N - 2$. In particular, the polyphase components of $x_i^{(d)}[n]$ are given by

$$x_i^{(d)}[2n] = x[n], \quad d = 0, 1; \quad (2)$$

$$x_i[2n+1] = \frac{1}{2}(x[n] + x[n+1]) + \frac{T}{8}(x^{(1)}[n] - x^{(1)}[n+1]) \quad (3)$$

$$x_i^{(1)}[2n+1] = \frac{2}{T}(x[n] + x[n+1]) - \frac{1}{2}(x^{(1)}[n] + x^{(1)}[n+1]) \quad (4)$$

where index n in (2) runs from 0 to $N - 1$, while in (3) and (4) runs from 0 to $N - 2$.

Once the interpolated sequences $x_i[n]$ and $x_i^{(1)}[n]$ have been obtained, the reconstruction of the function $x(t)$ within the interval $nT/2 < t < (n+1)T/2$ is given by

$$\hat{x}(t) = x_i[n] + x_i^{(1)}[n]\Delta t + \frac{x_i^{(1)}[n+1] - x_i^{(1)}[n]}{T}(\Delta t)^2 \quad (5)$$

where $\Delta t = t - nT/2$. It can be shown that this reconstruction can be implemented using the filter bank structure shown in Figure 1.

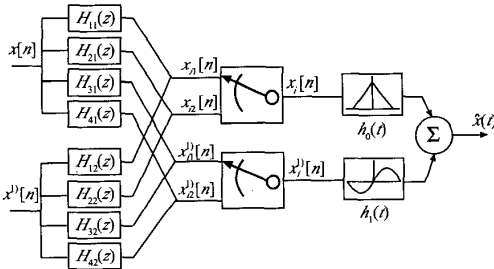


Fig. 1. Filter bank representation of the interpolation procedure

Here, the polyphase filters, $H_{ij}(z)$, can be easily obtained from (2)-(4), and the analog reconstruction filters are

$$h_0(t) = 1 - \frac{|t|}{T_i}, \quad \text{for } |t| \leq T_i. \quad (6)$$

$$h_1(t) = \frac{t}{2} \left(1 - \frac{|t|}{T_i} \right), \quad \text{for } |t| \leq T_i. \quad (7)$$

where $T_i = T/2$.

Obviously, the reconstructed signal, which is a spline function, can also be characterized in terms of a B-spline expansion from the sequence $x_i[n]$.

Moreover, it can be seen that the whole process of reconstruction is local, i.e., the value of $x(t)$ in $nT < t < (n+1)T$ depends only on $x^{(d)}[n]$ and $x^{(d)}[n+1]$, $d = 0, 1$. This fact makes the method directly applicable to nonuniform sampling.

This interpolation method can be extended to two-dimensional input spaces using the one-dimensional filter bank as basic reconstruction structure [7]. With this method it is possible to interpolate the partial derivatives of the function with respect to each variable of the input space and also the crossed derivatives.

2.1. Noise Sensitivity Analysis

In this section we analyze the degradation caused by the measurement noise in the interpolation procedure. We assume that the noise in the function and its derivative can be modeled as a zero-mean white Gaussian noise with variances σ_0^2 and σ_1^2 , respectively. Then, taking into account Eq. (2) to (5), it is easy to obtain the noise variance at any point of the reconstructed signals. Specifically, for $0 \leq t \leq T/2$, the noise variance in the reconstructed function $\sigma_{0r}^2(t)$ and its derivative $\sigma_{1r}^2(t)$ are given by

$$\sigma_{0r}^2(t) = \sigma_0^2 \left[\left(1 - \frac{2t^2}{T^2} \right)^2 + \frac{4t^4}{T^4} \right] + \sigma_1^2 \left[\left(1 - \frac{3t^2}{2T^2} \right)^2 + \frac{t^4}{4T^4} \right], \quad (8)$$

$$\sigma_{1r}^2(t) = \sigma_1^2 \left[\left(1 - \frac{3t}{T} \right)^2 + \frac{t^2}{T^2} \right] + \frac{32t^2}{T^4} \sigma_0^2. \quad (9)$$

The interesting point is that the noise variance for the derivative, $\sigma_{1r}^2(t)$, varies as $32\sigma_0^2 t^2 / T^4$. For instance, at $t = T/2$, the noise variance in the reconstructed derivative depends on $8\sigma_0^2 / T^2$. This points out the noise sensitivity of this method, mainly when the signals are over-sampled. This noise sensitivity is shared by all the methods employed to solve this interpolation problem, and it is the main reason to search for a regularization procedure.

3. PROPOSED REGULARIZATION TECHNIQUE

When the measurements are corrupted with noise, instead of requiring an exact interpolation a more adequate

alternative is to force some degree of smoothness in the solution. This can be achieved by minimizing the following functional

$$J = \sum_{d=0}^D \left(\lambda_d \sum_{n=0}^{N-1} (x_r^{(d)}[n] - x^{(d)}[n])^2 \right) + \lambda_r J_r \quad (10)$$

where $x_r^{(d)}[n]$ are the regularized sequences, which now do not fulfill the interpolation conditions (1). The first term measures the error of the solution with respect to the measurements, and the second one is the regularization term, that measures the smoothness of the solution. Here we will use one of the more usually employed measures of smoothness, which is the integral of the squared second derivative of the solution. For the particular case we are dealing with (function + first derivative, i.e., $D = 1$) it takes the following expression

$$J_r = \sum_{n=0}^{2N-3} (x_i^1[n] - x_i^1[n+1])^2. \quad (11)$$

Similarly to (2) and (4), the polyphase components of x_i^1 are now given by

$$x_i^1[2n] = x_r^1[n], \quad (12)$$

and

$$x_i^1[2n+1] = \frac{2}{T} (x_r[n] + x_r[n+1]) - \frac{1}{2} (x_r^1[n] + x_r^1[n+1]). \quad (13)$$

Therefore, substituting (12) and (13) into (11), we see that functional (10) only depends on the regularized sequences $x_r^{(d)}[n]$, $n = 0, \dots, N-1$. To obtain the minimum of (10) we evaluate its derivatives with respect to the components of the regularized sequences

$$\frac{\partial J}{\partial x_r[n]} = 2\lambda_0 (x_r[n] - x[n]) + \lambda_r \frac{\partial J_r}{\partial x_r[n]}, \quad (14)$$

$$\frac{\partial J}{\partial x_r^1[n]} = 2\lambda_1 (x_r^1[n] - x^1[n]) + \lambda_r \frac{\partial J_r}{\partial x_r^1[n]}. \quad (15)$$

Taking into account (12) and (13), it can be seen that for $n = 0$

$$\begin{cases} \frac{\partial J_r}{\partial x_r[n]} = \frac{16}{T^2} (x_r[n] - x_r[n+1]) \\ \quad + \frac{8}{T} (x_r^1[n] + x_r^1[n+1]); \\ \frac{\partial J_r}{\partial x_r^1[n]} = \frac{8}{T} (x_r[n] - x_r[n+1]) \\ \quad + 5x_r^1[n] + 3x_r^1[n+1]; \end{cases} \quad (16)$$

whereas for $0 < n < N-1$

$$\begin{cases} \frac{\partial J_r}{\partial x_r[n]} = \frac{16}{T^2} (2x_r[n] - x_r[n-1] - x_r[n+1]) \\ \quad + \frac{8}{T} (x_r^1[n+1] - x_r^1[n-1]); \\ \frac{\partial J_r}{\partial x_r^1[n]} = \frac{8}{T} (2x_r[n-1] - x_r[n+1]) + 10x_r^1[n] \\ \quad + 3x_r^1[n+1] + 3x_r^1[n-1]. \end{cases} \quad (17)$$

and, finally, for $n = N-1$

$$\begin{cases} \frac{\partial J_r}{\partial x_r[n]} = \frac{16}{T^2} (x_r[n] - x_r[n-1]) \\ \quad - \frac{8}{T} (x_r^1[n] + x_r^1[n-1]); \\ \frac{\partial J_r}{\partial x_r^1[n]} = \frac{8}{T} (x_r[n-1] - x_r[n]) \\ \quad + 5x_r^1[n] + 3x_r^1[n-1]; \end{cases} \quad (18)$$

Equating the derivatives (14) and (15) to zero, we obtain a linear system of equations

$$\mathbf{b} = \mathbf{A} \mathbf{x} \quad (19)$$

where

$$\mathbf{b} = [\lambda_0 (x[0], \dots, x[N-1]), \lambda_1 (x^1[0], \dots, x^1[N-1])]^T \quad (20)$$

$$\mathbf{x} = [x_r[0], \dots, x_r[N-1], x_r^1[0], \dots, x_r^1[N-1]]^T \quad (21)$$

and \mathbf{A} is an $2N \times 2N$ matrix. Typically, $\text{rank}(\mathbf{A}) = 2N$ and therefore the solution can be obtained as

$$\mathbf{x} = \mathbf{A}^{-1} \mathbf{b}. \quad (22)$$

Eq. (22) directly provides the regularized solution; however matrix inversion can be computationally expensive when the number of samples N is high. In this case, the problem can be solved more efficiently by using an iterative gradient-based algorithm. It is important to notice the local behavior of expressions (16) to (18), which makes that the computational burden of a gradient-based approach grows only linearly with N .

Once the regularized sequences have been obtained, the function is reconstructed using the filter bank structure presented in Figure 1.

We now comment how to select the appropriate values of the weight parameters λ_d , ($d = 0, 1$); and the regularization parameter λ_r . If there is not any knowledge about the noise power in the samples of the function (σ_0^2) and of the derivative (σ_1^2), we should fix $\lambda_0 = \lambda_1 = 1$ and select λ_r using cross-validation techniques. When it is possible to estimate these parameters, a simple alternative that provides suitable results, is to fix $\lambda_0 = 1$ and then select

$$\lambda_1 = \frac{\sigma_0^2}{\sigma_1^2}, \quad \lambda_r = \frac{P_0 \sigma_0^2}{P_1^2 T^2} \quad (23)$$

where P_0 and P_1 are the power of the function and the derivative respectively, which can be easily estimated.

Again, it is possible to extend this method of regularization to two-dimensional input spaces [7].

4. RESULTS

In this section we present some results obtained with the proposed regularized digital filtering method and compare its performance with the results provided by a non-regularized solution (interpolation), and by the extension of the Shannon sampling theorem using derivatives [4]. We have generated bandlimited signals as a linear combination of 100 sinusoids with random amplitudes, phases and frequencies. For this example, both signals $x(t)$ and $x^{(1)}(t)$ are sampled at four times its Nyquist rate, i.e., T is four times the inverse of the maximum frequency in Hz. The weight parameters are selected according to (23). Fig. 2 and 3 show the signal to noise ratio (SNR) for the reconstructed function and its derivative, respectively. The signal to noise ratio for the original sampled signal was $\text{SNR}_0=10$ dB, and we have considered different SNR_1 values for the sampled derivative. It can be seen that the regularized solution provides the best results, especially for the reconstruction of the derivative.

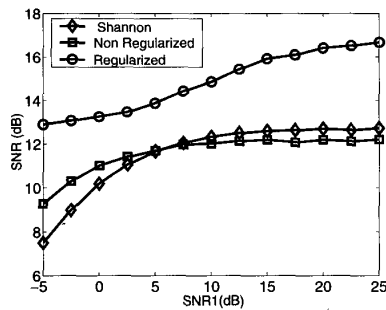


Fig. 2. Reconstruction of the function

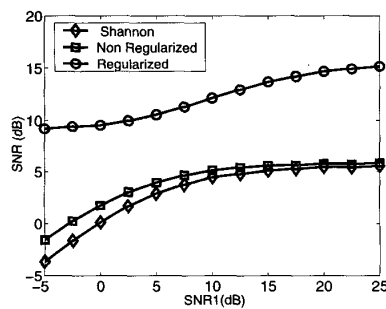


Fig. 3. Reconstruction of the first derivative

5. CONCLUSIONS

A new digital filtering method for the simultaneous reconstruction of a function and its derivatives has been

presented. The interpolation problem can be solved very efficiently using a local technique, which provides a solution in the space of polynomial splines and can be implemented by means of a polyphase filter bank. This interpolation technique, however, is very sensitive to noise. The regularization of the sampled sequences before reconstructing the signal considerably improves the performance of the proposed method in noisy environments. The improvement is more important when the sampling frequency increases, because the noise sensitivity of the interpolation method is higher in this case (see equations (8) and (9)). In particular, we have presented expressions for the case of a function and its first derivative. Nevertheless the model can be extended to higher order derivatives and irregular sampling. Some results have been presented showing the efficiency of this regularization method in noisy environments.

6. REFERENCES

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