COMPUTING THE DEGREES OF FREEDOM FOR ARBITRARY MIMO INTERFERENCE CHANNELS

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ABSTRACT

In this paper we provide an efficient procedure to compute the total number of degrees of freedom (DoF), achievable by linear beamforming, of the K-user multiple-input multiple-output (MIMO) interference channel with an arbitrary number of Tx-Rx antennas at each link. Firstly, we derive an analytical outer bound that generalizes the results that exist for the symmetric K-user $M \times N$ interference channel. Secondly, we obtain a tighter bound by solving a convex optimization problem that includes as constraints the DoF characterizations for point-to-point MIMO links and for 2-user interference channels. The solution to this convex problem admits an interesting waterfilling interpretation. Finally, exploiting this outer bound and using a recently proposed feasibility test, we show that it is possible to obtain the DoF for any interference channel in an efficient way. Some simulations results are included to illustrate the tightness of the derived bounds, as well as to study the DoF achievable for the 4-user channel when we distribute the total number of antennas among users and between transmitters and receivers in different ways.

Index Terms— Degrees of freedom, interference alignment, multiple-input multiple-output, convex optimization

1. INTRODUCTION

It has been recently shown that, to achieve the maximum spatial degrees of freedom (DoF) of the K-user multiple-input multiple-output (MIMO) interference channel (IC), the interference from other transmitters must be aligned at each receiver in a lower-dimensional subspace. This idea was first proposed for the MIMO X channel in [1, 2] and was thereafter called interference alignment (IA). It was shortly demonstrated that IA was instrumental to achieve the optimal $K/2$ DoF, when applied to the fully connected K-user single-input single-output channel with time-varying channel coefficients [3]. Spurred by this important result, many variants of the original alignment technique have recently been proposed to approximately characterize the capacity (in terms of DoF) of several multiterminal interference networks. In particular, in this paper we will focus on schemes that apply linear precoding and decoding in signal vector space for MIMO interference channels with constant channel coefficients.

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The 2-user MIMO IC was tackled in [4], leading to an exact expression for the DoF which is valid for both constant and varying channel coefficients. When $K \geq 3$ and the MIMO IC coefficients are constant, only some partial results exist. For example, in [5, 6, 7] the DoF of the 3-user $M \times N$ MIMO IC have been found for the case where every user sends the same number of streams. For the symmetric K-user IC with $M \times N$ links, inner and outer DoF bounds were found in [8, 9].

The problem of finding the DoF for an arbitrary IC is closely related to the problem of whether a given DoF tuple is feasible or not, which has recently given some interesting results for particular channel instances. Specifically, [5] proves a sufficient and necessary feasibility condition for the symmetric square case, where all links are $N \times N$ and there are at least three interfering users ($K \geq 3$). For the general case with arbitrary system parameters, only a necessary condition is proved in [5]. A general bound on the tuple of achievable DoF is given in [10], where it is also shown that, for the particular case of symmetric systems with a number of transmit and receive antennas multiple of the number of streams, the bound is tight. More recently, a fully general feasibility test has been proposed in [11, 12]. In principle, this test could be used to obtain the DoF for arbitrary K-user MIMO IC by exhaustively checking the feasibility of all possible DoF tuples. However, even for simple scenarios, this exhaustive search rapidly becomes intractable. In this paper, we address this problem and propose a much simpler procedure to find the DoF for arbitrary MIMO interference channels. As a first step, we propose several upper bounds with different degrees of complexity and tightness. Secondly, a search algorithm is proposed which, exploiting these bounds and using the feasibility test in [11, 12], is able to efficiently find the actual DoF for arbitrary networks.

2. SYSTEM MODEL

Consider the K-user MIMO IC comprised of K transmitter-receiver pairs. The k-th user is equipped with M_k and N_k antennas at the transmitter and receiver side, respectively, and each transmitter wishes to send $d_k \geq 0$ streams to the other side of the link. We adhere to the notation used in [13] and denote this general IC as $\prod_{k=1}^{K}(M_k \times N_k, d_k) = (M_1 \times N_1, d_1)\cdots(M_K \times N_K, d_K)$. The channel output at the k-th receiver is given by

$$\mathbf{y}_k = \mathbf{H}_{kk}\mathbf{V}_k\mathbf{s}_k + \sum_{l \neq k}\mathbf{H}_{kl}\mathbf{V}_l\mathbf{s}_l + \mathbf{n}_k, \quad (1)$$

where $\mathbf{s}_k$ is the $d_k \times 1$ transmitted signal, which is precoded by a full column-rank matrix $\mathbf{V}_k \in \mathbb{C}^{M_k \times d_k}$, and $\mathbf{n}_k$ is the zero-mean unit-variance circularly symmetric additive white Gaussian noise vector.

The first term in (1) is the desired signal, while the second term represents the interfering signal. In order to suppress the interference, the k-th receiver applies a linear decoder $\mathbf{U}_k \in \mathbb{C}^{N_k \times d_k}$, i.e.,

$$\mathbf{U}_k^H \mathbf{y}_k = \mathbf{U}_k^H \mathbf{H}_{kk} \mathbf{V}_k \mathbf{s}_k + \sum_{l \neq k} \mathbf{U}_k^H \mathbf{H}_{kl} \mathbf{V}_l \mathbf{s}_l + \mathbf{U}_k^H \mathbf{n}_k. \qquad (2)$$

The interference alignment problem consists in finding the matrices $\mathbf{V}_k$ and $\mathbf{U}_k$ in such a way that the interfering signals at each receiver fall into a reduced-dimensional subspace leaving an interference-free subspace for the desired signal. To this end, it is required that the polynomial equations

$$\mathbf{U}_k^H \mathbf{H}_{kl} \mathbf{V}_l = 0, \qquad k \neq l, \qquad (3)$$

are satisfied, while the signal subspace for each user must be linearly independent of the interference subspace and must have dimension d_k, that is

$$\operatorname{rank}(\mathbf{U}_k^H \mathbf{H}_{kk} \mathbf{V}_k) = d_k, \qquad \forall k \in \{1, \dots, K\}. \qquad (4)$$

In [13], systems were classified as either *proper* or *improper*. A system is *proper* if and only if for every subset of equations in (3), the number of variables is at least equal to the number of equations in that subset. This evaluation may be computationally demanding with the additional limitation that *properness* is necessary [10, 12] but not sufficient for a system to be feasible. For that reason, in this paper we will consider a simpler definition of *properness* which just considers the total set of equations, as in [14]. Let us define $s = \sum_k (M_k + N_k) d_k - \left(\sum_k d_k\right)^2 - \sum_k d_k^2$ as the difference between the number of variables and equations in the system (3). When $s < 0$ a system will be termed *improper*, and then infeasible, whereas a *proper* system ($s \geq 0$) can be either feasible or infeasible.

Finally, the capacity region $\mathcal{C}(\rho)$ of the K user MIMO IC is the set of all the achievable rate tuples $\mathbf{r}(\rho) \triangleq [r_1(\rho), \dots, r_K(\rho)]^T$ where $r_k(\rho)$ is the rate the k-th user can reliably sustain at a given signal to noise ratio (SNR), ρ. The sum capacity of the system, $\mathcal{C}_\Sigma(\rho)$, is the maximum sum rate achieved by any tuple in $\mathcal{C}(\rho)$. Its high SNR asymptote, $D \triangleq \lim_{\rho \to \infty} \frac{\mathcal{C}_\Sigma(\rho)}{\log \rho}$, is known as the DoF of the system, which, for our particular case, satisfies $D = \sum_{k=1}^K d_k$.

Hereinafter, we use the notation $\mathbf{w}$ for the column vector containing the variables w_k, i.e. $\mathbf{w} = [w_1, \dots, w_K]^T$. Additionally, we define $a_k = M_k + N_k$, $b_k = \min(M_k, N_k)$ and $\mathbf{J} = \mathbf{1}\mathbf{1}^T$. Throughout this paper we will use s in the following vector form:

$$s \triangleq \mathbf{a}^T \mathbf{d} - \mathbf{d}^T (\mathbf{J} + \mathbf{I}) \mathbf{d}. \qquad (5)$$

3. DOF OUTER BOUNDS

In this paper we are interested in obtaining the feasible tuple, $\mathbf{d}$, which maximizes the achievable DoF for an arbitrary MIMO-IC: $\prod_{k=1}^K (M_k \times N_k, d_k)$. More formally,

$$\begin{aligned} P_0: \quad & \underset{\mathbf{d}}{\text{maximize}} && \mathbf{1}^T \mathbf{d} && (6) \\ & \text{subject to} && \prod_{k=1}^K (M_k \times N_k, d_k) \text{ is feasible}, \\ & && \mathbf{d} \in \mathbb{N}^K. \end{aligned}$$

It is always possible to find the global optimizer of P_0 by exploring all possible DoF tuples, checking their feasibility by means of the test in [11], and selecting the tuple that maximizes $\mathbf{1}^T \mathbf{d}$. However, due to the combinatorial nature of the problem, this approach may be intractable. In order to diminish the associated computational cost, we will propose in the following three different relaxations of the original problem. These relaxations will allow us to find outer bounds for the DoF with an increasing degree of tightness.

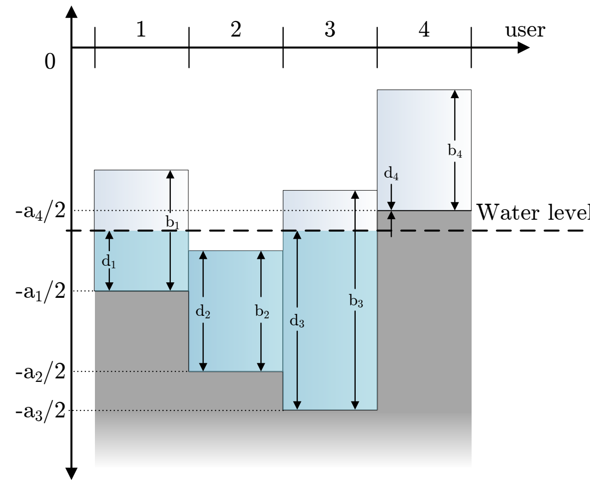


Fig. 1. Waterfilling interpretation of the proposed DoF bound. In this particular example, the point-to-point upper- and lower-bound constraints are active for users 2 and 4, respectively, yielding the tuple $d_1 = \frac{a_1}{2} + \mu$, $d_2 = b_2$, $d_3 = \frac{a_3}{2} + \mu$ and $d_4 = 0$, where μ is the water level.

3.1. Analytic bound

The first relaxation loosens the two constraints of the original problem, P_0. On the one hand, this new problem formulation replaces the feasibility condition by the *properness* condition as defined in Section 2, i.e. $s \geq 0$, where s is written as in (5). On the other hand, the entries of vector $\mathbf{d}$ are not required to be integer but real, i.e.,

$$\begin{aligned} P_1: \quad & \underset{\mathbf{d}}{\text{maximize}} && \mathbf{1}^T \mathbf{d} && (7) \\ & \text{subject to} && \mathbf{a}^T \mathbf{d} - \mathbf{d}^T (\mathbf{J} + \mathbf{I}) \mathbf{d} \geq 0, \\ & && \mathbf{d} \in \mathbb{R}^K. \end{aligned}$$

The objective function for this problem is an unbounded linear function of $\mathbf{d}$ and therefore its maximum is attained when the first constraint is active (i.e. $s = 0$). Consequently, it can be solved analytically by Lagrangian optimization. The global maximum for this problem, $\mathbf{1}^T \mathbf{d}^\star$, represents an upper bound of the original optimization problem (6), whose solution has to be necessarily an integer. Therefore, the bound is given by

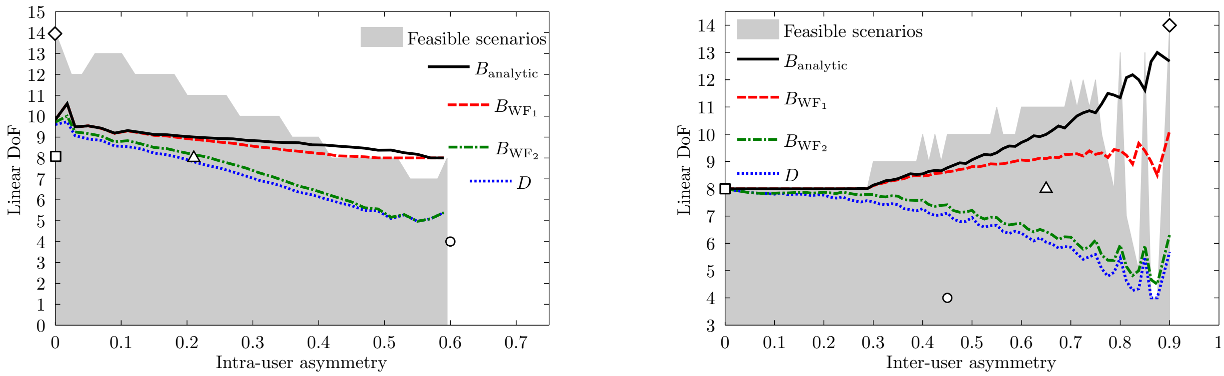
$$B_{\text{analytic}} = \left\lfloor \frac{\mathbf{1}^T \mathbf{a} + \sqrt{K(\mathbf{a}^T \mathbf{a}(K+1) - (\mathbf{1}^T \mathbf{a})^2)}}{2(K+1)} \right\rfloor. \qquad (8)$$

Remark 1: The expression in (8) can be trivially particularized to the case where all transmitter-receiver pairs have the same total number of antennas. Under this condition, (8) simplifies to

$$B_{\text{analytic}} = \left\lfloor \frac{\sum_{k=1}^K (M_k + N_k)}{K+1} \right\rfloor. \qquad (9)$$

It must be noticed that this expression generalizes the outer bound given in [15] for the K-user $M \times N$ MIMO interference channel, which is given by

$$D \leq \left\lfloor K \frac{M+N}{K+1} \right\rfloor. \qquad (10)$$



(a) Linear DoF vs. Intra-user asymmetry

(b) Linear DoF vs. Inter-user asymmetry

Fig. 2. Mean values of the proposed linear DoF bounds for different intra- and inter-user asymmetries. Some specific scenarios have been pointed for illustration: ○ $(2\times 8,1)^3(8\times 2,1)$, ◇ $(2\times 2,1)^3(14\times 14,11)$, □ $(5\times 5,2)^4$ and △ $(2\times 2,1)(3\times 5,1)(3\times 2,1)(7\times 16,5)$.

3.2. Waterfilling-based bounds

In order to improve the tightness of the analytical bound in (8), we can add additional constraints to the optimization problem. For instance, it is well-known that the number of streams transmitted by each of the users, when considered independently, has to satisfy the point-to-point bounds, $0 \le d_k \le b_k = \min(M_k, N_k)$. This consideration leads us to turn P_1 into

$$\begin{aligned} P_2: \quad & \underset{\mathbf{d}}{\text{maximize}} && \mathbf{1}^T\mathbf{d} \\ & \text{subject to} && \mathbf{a}^T\mathbf{d} - \mathbf{d}^T(\mathbf{J}+\mathbf{I})\mathbf{d} \ge 0, \\ &&& \mathbf{0} \le \mathbf{d} \le \mathbf{b}, \\ &&& \mathbf{d} \in \mathbb{R}^K. \end{aligned} \tag{11}$$

When formulated this way, the problem is convex and, hence, can be efficiently solved using standard software packages. Furthermore, in this case it is possible to obtain a waterfilling interpretation of the solution. To show this interpretation, let us first write the Lagrangian associated to the current optimization problem

$$L(\mathbf{d},\lambda,\boldsymbol{\alpha},\boldsymbol{\beta}) = \mathbf{1}^T\mathbf{d} + (\mathbf{a}^T\mathbf{d} - \mathbf{d}^T(\mathbf{J}+\mathbf{I})\mathbf{d})\lambda + \boldsymbol{\alpha}^T\mathbf{d} - \boldsymbol{\beta}^T(\mathbf{d}-\mathbf{b}). \tag{12}$$

Thus, the *Karush-Kuhn-Tucker* (KKT) conditions for this problem are

$$\begin{aligned} \mathbf{1} + (\mathbf{a} - 2(\mathbf{J}+\mathbf{I})\mathbf{d})\lambda + \boldsymbol{\alpha} - \boldsymbol{\beta} &= \mathbf{0}, \\ \mathbf{a}^T\mathbf{d} - \mathbf{d}^T(\mathbf{J}+\mathbf{I})\mathbf{d} \ge 0, \quad \mathbf{0} \le \mathbf{d} &\le \mathbf{b}, \\ \lambda \ge 0, \quad \boldsymbol{\alpha} \ge \mathbf{0}, \quad \boldsymbol{\beta} &\ge \mathbf{0}, \\ \lambda(\mathbf{a}^T\mathbf{d} - \mathbf{d}^T(\mathbf{J}+\mathbf{I})\mathbf{d}) &= 0, \\ \boldsymbol{\alpha} \circ \mathbf{d} = \mathbf{0}, \quad \text{and} \quad \boldsymbol{\beta} \circ (\mathbf{d}-\mathbf{b}) &= \mathbf{0}, \end{aligned} \tag{13}$$

where $\circ$ denotes the Hadamard product. From the first equation in (13), with $\boldsymbol{\alpha} = \mathbf{0}$ and $\boldsymbol{\beta} = \mathbf{0}$, the optimal distribution of streams among users can be written as

$$\mathbf{d}^\star = \left[\underbrace{\frac{1}{2(K+1)}\left(\frac{1}{\lambda^\star} - \mathbf{1}^T\mathbf{a}\right)\mathbf{1}}_{\text{Variable water level, } \mu} - \underbrace{\left(-\frac{\mathbf{a}}{2}\right)}_{\text{Floor}} \right]_{\mathbf{0}}^{\mathbf{b}}. \tag{14}$$

where $[\bullet]_{\boldsymbol{\ell}}^{\mathbf{u}}$ is the element-wise operator $\min(\mathbf{u}, \max(\boldsymbol{\ell}, \bullet))$. Therefore, (12) admits a waterfilling interpretation which is as follows (see Figure 1):

1. For each of the K users, set up a unit-base vessel with height b_k on top of a floor of height $-a_k/2$.
2. Pour water keeping a flat water level across all vessels. The available volume of water is given by s. In other words, $s < 0$ means that you have exceeded the total amount of water.
3. If some vessel overflows, keep filling the rest of vessels.
4. When all the available water has been poured or all the vessels have been filled, the amount of water in each vessel gives the optimum stream value for that user, $d_k^\star$.

The total amount of water, $\sum_k d_k^\star = \mathbf{1}^T\mathbf{d}^\star$, gives us the sought DoF upper bound, $B_{\text{WF}_1} = \lfloor \mathbf{1}^T\mathbf{d}^\star \rfloor$.

Remark 2: The foregoing interpretation shows that the optimal (non-integer) stream profile is a downshifted version of $\mathbf{a}/2$ (the mean number of antennas per user), which is element-wise bounded from the top and the bottom by $\mathbf{b}$ and $\mathbf{0}$, respectively.

The previous bound can be further improved by adding, as constraints, the DoF results for the 2-user IC obtained in [4]. In particular, any two users in the channel must satisfy

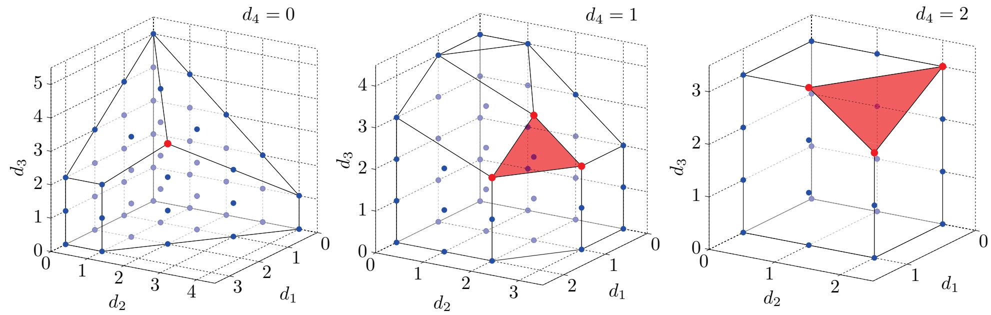
$$d_i + d_j \le \min(N_i + N_j, M_i + M_j, \max(N_i, M_j), \max(N_j, M_i)). \tag{15}$$

In addition to include all pairs of users, we can also consider groups of users that cooperate and jointly process their data in such a way that a new 2-user interference channel is created. In summary, considering all cooperative 2-user groups gives us a total of $2^K - K - 1$ new upper bounds that can be added to our optimization problem. When all these bounds along with the point-to-point ones are represented by the variables $b_{\mathcal{J}}, \ \forall \mathcal{J} \subseteq \mathcal{K}$, they lead to a convex optimization problem similar to P_2, but with the second constraint substituted by:[1]

$$0 \le \sum_{k \in \mathcal{J}} d_k \le b_{\mathcal{J}}, \quad \forall \mathcal{J} \subseteq \mathcal{K}. \tag{16}$$

Once the optimal solution to this problem is found, a new upper bound of the DoF is obtained by taking the largest previous integer $B_{\text{WF}_2} = \lfloor \mathbf{1}^T\mathbf{d}^\star \rfloor$.

[1]This problem admits a similar waterfilling interpretation, but the details are omitted due to lack of space.



(a) Linear DoF region for $d_4 = 0$ (b) Linear DoF region for $d_4 = 1$ (c) Linear DoF region for $d_4 = 2$

Fig. 3. DoF region for the 4-user system $(3 \times 4, d_1)(4 \times 11, d_2)(5 \times 5, d_3)(6 \times 2, d_4)$ using linear beamforming.

4. COMPUTING THE DOF

In this section we propose an efficient algorithm to find the actual linear DoF value. The algorithm is basically an ordered search that starts from those tuples whose DoF are exactly any of the outer bounds obtained in this paper and works as follows:

1. Assume that the DoF are bounded above by B (which can be any of the bounds in this paper).
2. Generate all the possible tuples of integer numbers adding up exactly B. In number theory this is usually referred to as computing all the restricted compositions of an integer B. The term *restricted* comes from two facts: 1) the number of summands or *parts* that the composition is allowed to have is equal to K; 2) there exist upper and lower bounds on the values of each part. More specifically, each part, d_k, must verify $0 \leq d_k \leq b_k$. A great deal of attention has been focused on algorithms that are able to compute restricted compositions, leading to reasonably fast algorithms with good time complexity as the one in [16] ($O(K)$ per composition where K equals the number of *parts*). For the interested reader, the work [16] also provides closed form solutions to the problem of counting these doubly restricted integer compositions.
3. Check only those *proper* tuples for feasibility by means of the test in [11]. B is the DoF value as soon as a feasible tuple is found. If no feasible tuple is found among them, then decrease the value of the bound, $B := B - 1$, and start over again.

5. SIMULATION RESULTS

In this section we show several simulation results that illustrate the ideas presented in this paper. As an example, we consider a 4-user MIMO IC with a total number of antennas: $\sum_{k=1}^{K}(M_k + N_k) = 40$, and with $M_k \geq 2$ and $N_k \geq 2$, $\forall k$. We study the DoF that can be achieved with linear beamforming for different distributions of the total number of antennas among users and between transmitters and receivers. Two different measures are proposed to quantify the asymmetry of a given scenario:

$$\underbrace{\frac{1}{K}\sum_k \frac{|M_k - N_k|}{M_k + N_k}}_{\text{Intra-user asymmetry}}, \quad \text{and} \quad \underbrace{\frac{1}{K}\sum_k \frac{|M_k - \bar{M}| + |N_k - \bar{N}|}{\bar{M} + \bar{N}}}_{\text{Inter-user asymmetry}}, \tag{17}$$

where $\bar{M}$ and $\bar{N}$ are the mean number of transmit and receive antennas, respectively. Notice that there can be more than one network with the same value of inter- and intra-user asymmetry, therefore we will depict mean DoF values (averaged over all networks with the same level of asymmetry) in the plots.

Figures 2(a) and 2(b) show how the mean value of the proposed bounds evolves with the intra- and inter-user asymmetry, respectively. The mean value of the actual DoF, D, is also shown in a dotted line while the shaded area represents the whole range of feasible DoF values and, thus, it may exceed the mean value of the upper bounds. Both figures show that the waterfilling bound that incorporates the 2-user constraints (denoted as B_{WF_2}) is very close to the actual achievable DoF in the whole asymmetry range. On the other hand, the analytic bound and the waterfilling bound that uses only point-to-point constraints (B_{WF_1}), are only tight for low values of asymmetry. This result suggests that for highly asymmetric networks, the DoF are mainly limited by the 2-user channel constraints, whereas in symmetric or close-to-symmetric IC the *properness* condition limits the DoF. Alternatively, Fig. 2(a) shows that the maximum DoF decreases as the intra-user asymmetry increases, and so does in the mean sense. However, as Fig. 2(b) shows, when the inter-user asymmetry increases the maximum DoF increases as well, although the mean DoF value decreases. Let us remind that, for a given value of any of the asymmetries, there are different scenarios that achieve different DoF. For the sake of illustration, we have pointed some specific scenarios in both figures. One such example is the system $(2 \times 2, 1)^3(14 \times 14, 11)$ (designated by $\diamond$), which, in spite of having a high inter-user asymmetry, is able to achieve a total of 14 DoF with minimum intra-user asymmetry.

As a final example, Figure 3 shows the DoF region of the system $(3 \times 4, d_1)(4 \times 11, d_2)(5 \times 5, d_3)(6 \times 2, d_4)$, which has been obtained by checking the feasibility of all the possible DoF tuples. It required a total of 208 executions of the feasibility test, whereas computing the tuples in red which satisfy $\sum_k d_k = 7$ and achieve the DoF of the system, required only 7 executions when initialized in B_{WF_2}.

6. CONCLUSION

In this paper, we have derived three different bounds on the spatial DoF of an arbitrary MIMO IC. Based on these bounds and on a recently proposed feasibility test, we have designed an efficient procedure to compute the achievable DoF of a given scenario. Moreover, we have shown that two of these bounds admit an interesting waterfilling interpretation. Our simulations illustrate the tightness of the proposed bounds for a wide variety of scenarios. Further improvements on these bounds represent our current lines of future research.

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