BAYESIAN ESTIMATION OF A CLASS OF CHAOTIC SIGNALS

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ABSTRACT

Chaotic signals are potentially atractive in a wide range of signal processing applications. This paper deals with Bayesian estimation of chaotic sequences generated by tent maps and observed in white noise. The existence of invariant distributions associated with these sequences makes the development of Bayesian estimators quite natural. Both Maximum a Posteriori (MAP) and Minimum Mean Square Error (MS) estimators are derived. Computer simulations confirm the expected performance of both approaches and show how the inclusion of a-priori information produces in most cases an increase in performance over the Maximum Likelihood (ML) case.

1. INTRODUCTION

Chaotic signals, signals generated by a non-linear dynamical system in chaotic state, have received much attention in the past years. Classical signal processing techniques are not adequate for this class of signals that, while deterministic, exhibit a noise-like behaviour. Consequently it is important to develop new algorithms matched to this type of signals. In particular there is a need for robust and efficient algorithms for the estimation of these signals in noise.

Estimation of chaotic signals has been addressed in several papers. In [1] the performance of the ML estimator for a chaotic signal is analyzed. It is found that the estimator is inconsistent, so the asymptotic distribution for large data records is invalid. However for a high Signal to Noise Ratio (SNR) the ML estimator is asymptotically unbiased and attains the Cramer-Rao lower bound (CRLB). In [2] an algorithm for chaotic signal estimation based on the connection between the symbolic sequence and the initial condition is presented, which is shown to achieve the CRLB at high SNR. This approach is closely related with the "halving method" presented in [3] where a dynamical programming ML estimator is also proposed. Finally in [4] a recursive implementation of the ML estimator of chaotic signals generated by tent maps is derived.

In this paper we develop Bayesian estimators for chaotic signals whose dynamics are governed by the tent map. The selection of the a-priori density is based on the invariant density associated with chaotic sequences generated by a tent map. To develop the Bayesian estimators, a novel implementation of the ML estimator is proposed, necessary to derive a closed form expression for the posterior density. The resulting MAP and MS estimates show good performance at low SNR in comparison with ML estimates, are asymptotically unbiased and achieve the CRLB for high SNR.

2. TENT MAPS

The signals $x[n]$ that we consider in this work are generated according to:

$$x[n] = F(x[n-1]) \tag{1}$$

where $F(.)$ is a symmetric tent map

$$F(x) = \beta - 1 - \beta\,|x| \tag{2}$$

for some parameter $1 < \beta \leq 2$.

These maps produce sequences that are chaotic, and all the points in the interval $(-1, \beta-1]$ tend to the attracting interval $\left[-(\beta-1)^2, \beta-1\right]$ [5]. However their invariant densities, cannot, in general, be readily described in closed form, except for the map corresponding to $\beta = 2$, where the invariant density is uniform:

$$p(x) = \begin{cases} 0.5 & |x| < 1 \\ 0 & \text{Otherwise} \end{cases} \tag{3}$$

The function $F(.)$ is noninvertible, but is also unimodal and even, so it has two inverse images that differ only in the sign. The two inverses of $F(.)$ can be denoted

$$F_s^{-1}(v) = \frac{\beta - 1 - v}{\beta} s \tag{4}$$

This work has been supported by CICYT grant TIC96-0500-C10-07

where $s = \pm 1$. Therefore

$$v = F(x) \Rightarrow x = F^{-1}_{sign(x)}(v)$$

A useful representation for our chaotic sequence can be obtained via inverse mappings. Any element of the sequence can be obtained by backward iteration from $x[N]$

$$x[n] = F^{-1}_{s[n]} \circ F^{-1}_{s[n+1]} \circ \cdots \circ F^{-1}_{s[N-1]} (x[N]) \qquad (5)$$

with $s[n] = sign(x[n])$. This sign sequence (also called itinerary) $\mathbf{s} = s[0], ..., s[N-1]$, can also be considered a symbolic coding of the chaotic signal [2]. Several estimation methods [2][3] rely on the fact that this itinerary defines a region in the phase space where the initial condition must lie. This fact will also arise in the deduction of Bayesian estimators. We will denote R_i the region of the phase space associated with a certain itinerary $\mathbf{s}_i$. The length of R_i will be denoted Δ_i and d_i its middle point. Finally we will define an indicator (sometimes called characteristic) function

$$\chi_i(x) = \begin{cases} 1 & if\ x \in R_i \\ 0 & if\ x \notin R_i \end{cases} \qquad (6)$$

When $\beta = 2$ there are 2^N regions, each of size $\Delta = 2^{(1-N)}$. In any other case the number of regions, that we will denote P, is less than 2^N. This means that some itineraries will not be possible, and this a-priori knowledge will improve the performance of Bayesian approaches.

To derive the Bayesian estimators we need a closed form expression for $x[n]$ as a function of the initial condition and the itinerary

$$x[n] = F^n_{\mathbf{s}}(x[0]) = (\beta - 1) \sum_{i=0}^{n-1} S^n_i \beta^i + S^n_n \beta^n x[0] \qquad (7)$$

where $S^n_0 = 1$ and

$$S^n_i = (-1)^i \prod_{j=n-i}^{n-1} s[j] \qquad i = 1, \ldots, n \qquad (8)$$

and $F^n_{\mathbf{s}}(x[0])$ is the n fold composition of F. The index $\mathbf{s}$ stresses the fact that this expression is only true if $x[0]$ has a sign sequence given by $\mathbf{s}$.

3. BAYESIAN ESTIMATION OF TENT MAP SEQUENCES

3.1. Problem Statement

The data model for the problem we are considering is

$$y[n] = x[n] + w[n] \qquad n = 0, 1, \ldots, N \qquad (9)$$

where $w[n]$ is a stationary, zero-mean white Gaussian noise with variance σ^2, and $x[n]$ is generated using (2) by iterating some unknown $x[0] \in \left[-(\beta-1)^2, \beta - 1\right]$ according to (1) for some parameter $1 < \beta \leq 2$. In this paper we will address Bayesian estimation of the initial condition $x[0]$. The rest of the signal components may be estimated in a similar way.

3.2. ML estimation of Tent Map Sequences

ML signal estimation produces the initial condition that minimizes

$$J(x[0]) = \sum_{k=0}^{N} \left(y[k] - F^k(x[0])\right)^2 \qquad (10)$$

In [4] a recursive ML signal estimator for chaotic sequences generated by tent maps is derived. The algorithm makes use of the sign sequence associated with the initial condition and is partitioned in two stages: filtering and smoothing. The filtering stage provides forward estimates $\hat{x}[n \mid n]$ of $x[n]$

$$\hat{x}[n\,|n] = \frac{\left(\beta^2 - 1\right) \beta^{2n} y[n] + \left(\beta^{2n} - 1\right) \hat{x}[n\,|n-1]}{\left(\beta^{2(n+1)} - 1\right)} \qquad (11)$$

This filtering stage is complemented with hard censoring when the forward estimates lie outside the admissible interval to obtain the forward ML estimates $\hat{x}_{ML}[n \mid n]$. The itinerary is given by

$$\hat{s}[n\,|n] = \hat{s}[n] = sign(\hat{x}[n\,|n]) \qquad (12)$$

As $\hat{x}_{ML}[N\,|N] = \hat{x}_{ML}[N]$ is the ML estimate of $x[N]$, and making use of the invariance property of the ML estimator, the rest of the sequence estimates may be obtained by iterating backwards from $\hat{x}_{ML}[N]$

$$\hat{x}_{ML}[n] = F^{-1}_{\hat{s}[n]} \circ F^{-1}_{\hat{s}[n+1]} \circ \cdots \circ F^{-1}_{\hat{s}[N-1]} (\hat{x}_{ML}[N]) \qquad (13)$$

Using (7), we can express (10) in a certain region R_i

$$J_i(x[0]) = \sum_{k=0}^{N} \left(y[k] - F^k_{\mathbf{s}_i}(x[0])\right)^2 \qquad (14)$$

and differentiating and solving for the unique minimum we obtain an estimate of $x[0]$ for a given itinerary

$$\hat{x}_i[0] = \frac{\sum\limits_{k=0}^{N} \left(y[k] - (\beta - 1) \sum\limits_{i=0}^{k-1} S^k_i \beta^i\right) S^k_k \beta^k}{\sum\limits_{k=0}^{N} \beta^{2k}} \qquad (15)$$

This is the ML estimate of $x[0]$ with an known itinerary given by $\mathbf{s}_i$ when $\hat{x}_i[0] \in R_i$. Otherwise, it is easily proved that the ML estimate is the closest point to $\hat{x}_i[0]$ in R_i. Given (14) we can write (10) as

$$J(x[0]) = \sum_{i=1}^{P} \chi_i(x[0]) J_i(x[0]) \tag{16}$$

3.3. Prior Density

To develop the Bayesian estimators, we need to define the prior density for the initial condition $x[0]$. The obvious choice is to assign the invariant density associated with the known map.When $\beta = 2$ the invariant density is uniform and given by (3). For other values, however, no closed form expressions are available. To circunvent this problem, we propose to express the prior densities according to

$$p(x[0]) = \sum_{i=1}^{P} p_i \chi_i(x[0]) \tag{17}$$

where the p_i constants are given by

$$p_i = \frac{1}{\Delta_i} \int_{R_i} p(x) dx \tag{18}$$

The invariant density is substituted by a staircase approximation. The constants p_i may be estimated from long sequences of data generated according to the model (1) (2) with the selected value of β.

3.4. Posterior Density

As our observations $\mathbf{y} = y[0], y[1], ..., y[N]$ are a collection of independent Gaussian random variables with equal variance, and using (16), the conditional density may be expressed

$$p\left(\mathbf{y}\,|x[0]\right) = \frac{1}{\gamma^{N+1}} \sum_{i=1}^{P} \chi_i(x[0]) \exp\left(-\frac{J_i(x[0])}{2\sigma^2}\right) \tag{19}$$

with $\gamma = \sqrt{2\pi}\sigma$. Applying the Bayes rule, the posterior density becomes

$$p\left(x[0]\,|\mathbf{y}\right) = K \sum_{i=1}^{P} p_i \chi_i(x[0]) \exp\left(-\frac{J_i(x[0])}{2\sigma^2}\right) \tag{20}$$

It is clear from (7) that (14) is a quadratic function of x[0]. So, by completing the square using (15) and after some straightforward calculations, we obtain

$$p\left(x[0]\,|\mathbf{y}\right) = K \sum_{i=1}^{P} a_i \chi_i(x[0]) \exp\left(-\frac{(x[0] - \hat{x}_i[0])^2}{2\sigma_p^2}\right) \tag{21}$$

with

$$a_i = p_i \exp\left(-\frac{J_i(\hat{x}_i[0])}{2\sigma^2}\right) \tag{22}$$

and

$$\sigma_p^2 = \sigma^2 \left(\sum_{i=0}^{N} \beta^{2i}\right)^{-1} \tag{23}$$

Finally, integrating (21)

$$K^{-1} = \sum_{i=1}^{P} a_i A_i \tag{24}$$

with

$$A_i = \int_{R_i} \exp\left(-\frac{(x[0] - \hat{x}_i[0])^2}{2\sigma_p^2}\right) dx[0] \tag{25}$$

3.5. Bayesian Estimators

The conditional density given by (21) is composed of P truncated Gaussians weighted by the coefficients given by (22). The maximum of each Gaussian is given by the ML estimate for a known itinerary

$$\hat{x}^i_{ML}[0] = \begin{cases} \hat{x}_i[0] & \hat{x}_i[0] \in R_i \\ d_i + 0.5\Delta_i & \hat{\xi}_i > 0.5\Delta_i \\ d_i - 0.5\Delta_i & \hat{\xi}_i < 0.5\Delta_i \end{cases} \tag{26}$$

where $\hat{\xi}_i = \hat{x}_i[0] - d_i$ is the displacement of each estimate from the midpoint of the associated region. Given k, the index of $x^i_{ML}[0]$ that maximizes

$$k = \arg\max_i \left(p_i \exp\left(-\frac{J(\hat{x}^i_{ML}[0])}{2\sigma^2}\right)\right) \tag{27}$$

then $\hat{x}_{MAP}[0] = \hat{x}^k_{ML}[0]$.

The MS estimate is the mean of the posterior density given by (21). The resulting estimator is

$$\hat{x}_{MS}[0] = \frac{\sum_{i=1}^{P} a_i A_i \left(\hat{x}_i[0] - \frac{\sigma_p}{\sqrt{2\pi}A_i} g(\hat{\xi}_i)\right)}{\sum_{i=1}^{P} a_i A_i} \tag{28}$$

where

$$g(\hat{\xi}_i) = \exp\left(\frac{\left(\hat{\xi}_i - 0.5\Delta_i\right)^2}{-2\sigma_p^2}\right) - \exp\left(\frac{\left(\hat{\xi}_i + 0.5\Delta_i\right)^2}{-2\sigma_p^2}\right) \tag{29}$$

Both Bayesian estimators clearly resemble the well known MAP and MS estimators for a constant signal in Gaussian noise with uniform prior density [6]. If we consider the case $\beta = 2$, all the p_i and A_i are equal and the expressions simplify considerably. In particular it is easy to verify that the MAP estimate is equal to the ML estimate in this case.

SNR	$-10 * log10(MSE)$					
	$N = 4$			$N = 8$		
	ML	MAP	MS	ML	MAP	MS
0 dB	13.5	15.7	18.1	13.5	15.7	18.5
5 dB	20.0	20.9	22.7	19.9	20.7	22.6
10 dB	27.3	27.5	29.0	27.3	27.4	29.0
15 dB	34.3	34.3	35.8	34.7	34.8	36.6
20 dB	42.0	42.0	43.4	42.4	42.4	44.3
25 dB	48.9	48.9	50.2	51.0	51.0	53.2
60 dB	87.5	87.5	87.8	95.5	95.5	97.5

Table 1: Average MSE in dB of the three estimators.

4. SIMULATION RESULTS

In this section we analyse the performance of the Bayesian estimators in comparison with the ML estimator. We consider a map with $\beta = 1.5$. 1000 initial points have been selected according to the invariant density of the map and iterated to obtain length $N+1$ records with $N = 4$ and $N = 8$. Table 1 shows the estimators Mean Square Error (MSE) obtained by Monte Carlo simulations by averaging 1000 cases for each initial condition and SNR. Both Bayesian estimators improve the ML estimator performance for low SNR and the three of them are asymptotically unbiased and attain the CRLB at high SNR. Due to the inconsistency of the ML estimator there is very little increase in performance by doubling N except at high SNR (60 dB in this case). An example is shown in Fig. 1, also with $\beta = 1.5$.

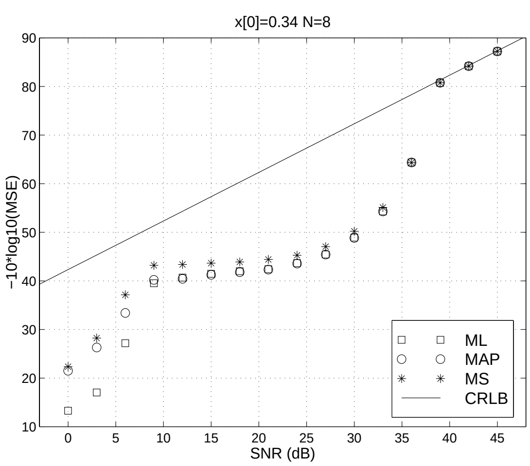

Figure 1: Mean square error of the three estimators.

Finally, a brief comment about computational cost. This cost depends on the number of regions P, which is 2^N for $\beta = 2$, but in general is much lower. The number of regions for $\beta = 1.5$ is shown in Table 2.

$\beta = 1.5$							
N	3	4	5	6	7	8	9
P	5	7	11	16	25	37	57

Table 2: Number of regions as a function of N

5. CONCLUSIONS

In this work we have developed Bayesian estimators for a class of chaotic signals derived from one-dimensional nonlinear maps. We have introduced a new implementation of the ML estimator and, using it, we have developed a closed form expression for the posterior density of the initial condition using the invariant density of the map as prior density. The Bayesian estimators have shown to improve the ML performance at low SNRs. The computational cost, although increased over the ML estimator, remains reasonable for moderate sized data records.

Further lines of research include developing Bayesian estimators for other classes of chaotic maps such as markov maps, searching for new performance bounds that better capture the performance of optimal estimators and looking for suboptimal estimation approaches that reduce the computational cost by only considering a small subset of the possible itinieraries.

6. REFERENCES

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