# Interference Alignment in MIMO Networks: Feasibility and Transceiver Design 

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## Outline



## Interference management in wireless networks

- Weak interference: treat as noise
- Strong interference: decode
- Interference strength comparable to signal strength:
- Traditional approach: orthogonalize (TDMA, FDMA,...)

Each user gets $\frac{1}{K}$ of channel resources ( $\frac{1}{K} D o F$ )

- Interference alignment [Cadambe \& Jafar, 2008]:

Every user gets $\frac{1}{2}$ of channel resources ( $\frac{1}{2}$ DoF)

- Users cooperate so that interfering signals overlap at each receiver, leaving more room for desired signals
- Achieves many more DoF than previously believed
- Ideally, all interfering users are jointly perceived as a single one

Everyone gets half the cake!

## Outline



## Feasibility of linear spatial domain IA: problem statement

 Arbitrary K-user interf. channel: $\left(M_{1} \times N_{1}, d_{1}\right) \cdots\left(M_{K} \times N_{K}, d_{K}\right)$ Interference alignment conditions:

$$
\begin{gathered}
\mathbf{U}_{k}^{T} \mathbf{H}_{k l} \mathbf{V}_{l}=\mathbf{0}, \quad k \neq I, \\
\operatorname{rank}\left(\mathbf{U}_{k}^{T} \mathbf{H}_{k k} \mathbf{V}_{k}\right)=d_{k}, \forall k .
\end{gathered}
$$

Assumptions:

- No channel extensions allowed
- Generic choice of channel matrices

Feasibility problem: determine if there exists (at least) a set of precoders, $\left\{\mathbf{V}_{l}\right\}$, and decoders, $\left\{\mathbf{U}_{k}\right\}$, which satisfies the above set of bilinear equations

## Feasibility of linear spatial domain IA

Existing results

- Conclusive answer for certain symmetric scenarios, e.g., $(M \times N, d)^{3},(M \times M, d)^{K}$ or $(M \times N, d)^{K}$ where $d \mid M$ and $N$
- Results for asymmetric scenarios have remained elusive

Theorem (Razaviyayn et al., 2012): Any DoF tuple ( $d_{1}, d_{2}, \ldots, d_{K}$ ) that is achievable through IA must satisfy the following

$$
\min \left(M_{k}, N_{k}\right) \geq d_{k}, \quad \forall k
$$

$$
\max \left(M_{l}, N_{k}\right) \geq d_{l}+d_{k}, \quad \forall(k, l) \in \Phi
$$



Remark: Costly evaluation for a necessary (but not sufficient) condition. Only sufficient when $d \mid M$ and $N$, e.g., single-beam

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\begin{aligned}
& \text { P2P conditions for every user: } O(K) \\
& \max \left(M_{l}, N_{k}\right) \geq d_{l}+d_{k}, \quad \forall(k, I) \in \Phi
\end{aligned}
$$

$\sum_{(k, l) \in \phi}\left(M_{l}-d_{l}\right) d_{l}+\sum_{k:(k, l) \in \phi}\left(N_{k}-d_{k}\right) d_{k} \geq \sum_{(k, l) \in \phi} d_{l} d_{k} \forall \phi \subseteq \phi$
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Signal+Interf. accommodated at TX or RX: $O\left(K^{2}\right)$

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Signal+Interf. accommodated at TX or RX: $O\left(K^{2}\right)$
\#vars. $\geq$ \#eqs. for every subset of eqs.: $O\left(2^{K^{2}}\right)$

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$$

Signal+Interf. accommodated at TX or RX: $O\left(K^{2}\right)$
Properness [Yetis et al., 2010]: $O\left(2^{K^{2}}\right)$

Remark: Costly evaluation for a necessary (but not sufficient) condition. Only sufficient when $d \mid M$ and $N$, e.g., single-beam

## First contribution

## Feasibility solved in polynomial time (for scenarios where properness is also sufficient)

Sometimes, more efficiently, e.g. single-beam $\rightarrow$ linear time

- How? Identify the properness conditions with conditions for existence of a feasible flow in a supply-demand network



## Example: $(4 \times 2,1)(2 \times 2,1)^{2}(2 \times 4,1)$ system



1. Calculate the maximum flow with any polynomial time algorithm: Ford-Fulkerson, Edmonds-Karp, Goldberg
2. Check demand fulfillment

$$
\text { Demand fulfillment }=\text { Feasibility }
$$

(for those scenarios where properness and feasibility are equivalent)

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We still need a general result!

## Second contribution: a general feasibility test

We need an appropriate mathematical model considering problem invariances
$(H, U, V)$
$\mathcal{V}$
$\pi_{1}$

$\mathcal{H}$$\quad \swarrow \quad \searrow$| $\pi_{2}$ |
| :--- |
| $\mathcal{S}$ | (H)

Algebraic approach [Razaviyayn et al., 2012; Bresler et al., 2014]
Case 1: \#vars $<\#$ eqs or $\operatorname{dim}(\mathcal{V})<\operatorname{dim}(\mathcal{H})$


- $\pi_{1}(\mathcal{V})$ cannot cover most of $\mathcal{H}$ :

No solution for every choice of H out of a zero-measure set

Case 2: \#vars $\geq \#$ eqs or $\operatorname{dim}(\mathcal{V}) \geq \operatorname{dim}(\mathcal{H})$


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Case 2: \#vars $\geq \#$ eqs or $\operatorname{dim}(\mathcal{V}) \geq \operatorname{dim}(\mathcal{H})$

1. $\pi_{1}(\mathcal{V})$ covers the whole $\mathcal{H}$ : Solution for every choice of H
2. $\pi_{1}(\mathcal{V})$ is a zero-measure set of $\mathcal{H}$

No solution for every choice of H out of a zero-measure set
The whole $\mathcal{V}$ projects onto $\mathcal{H}$ in a singular way

## How to distinguish Case 2.2 from Case 2.1? In Case 2.2...

## Tangent space of $\mathcal{V}$ does not project onto the whole tangent space of $\mathcal{H}$

- Formally, every point of $\mathcal{V}$ is a critical point of $\pi_{1}$
- By definition, the derivative of $\pi_{1}$ at critical points is not surjective, which is equivalent to $\theta$, i.e.

$$
\left(\dot{U}_{1}, \ldots, \dot{U}_{K}, \dot{V}_{1}, \ldots, \dot{V}_{k}\right) \mapsto\left\{\dot{U}_{k}^{T} H_{k l} V_{l}+U_{k}^{T} H_{k l} \dot{V}_{l}\right\}
$$

being not surjective (rank deficient)

- Tools from differential topology (Ehresmann's Theorem) prove
- $\theta$ is almost everywhere surjective (feasible)
- $\theta$ is nowhere surjective (infeasible)

> It is enough to test for surjectivity at some affine representative, $\left(\dot{\mathbf{U}}_{1}, \ldots, \dot{\mathbf{U}}_{K}, \dot{\mathbf{V}}_{1}, \ldots, \dot{\mathbf{V}}_{K}\right)$, of a vector in the tangent space of $\mathcal{V}$

## Main result

Theorem: An IA scenario is feasible iff for almost every choice of $\mathbf{H}_{k l}$, and for any choice of $\mathbf{U}_{k}, \mathbf{V}_{l}$ satisfying the IA conditions, the linear mapping defined by

$$
\left(\dot{\mathbf{U}}_{1}, \ldots, \dot{\mathbf{U}}_{k}, \dot{\mathbf{V}}_{1}, \ldots, \dot{\mathbf{V}}_{k}\right) \mapsto\left\{\dot{\mathbf{U}}_{k}^{T} \mathbf{H}_{k l} \mathbf{V}_{l}+\mathbf{U}_{k}^{T} \mathbf{H}_{k l} \dot{\mathbf{V}}_{l}\right\}
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is surjective.

Affine representatives (matrices) $\Rightarrow$ Linear Algebra

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It is easy to devise a simple numerical feasibility test:

- Step 1: Find an arbitrary IA solution


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Affine representatives (matrices) $\Rightarrow$ Linear Algebra
It is easy to devise a simple numerical feasibility test:

- Step 1: Find an arbitrary IA solution
- Step 2: Rank checking


## Step 1: Find an arbitrary IA solution

- We need to find an arbitrary set $\left(\mathbf{H}_{k l}, \mathbf{U}_{k}, \mathbf{V}_{l}\right)$ such that the IA conditions are satisfied
- The following canonical representatives trivially satisfy the alignment conditions:
- Precoders/decoders:

$$
\mathbf{V}_{l}=\left[\begin{array}{c}
\mathbf{I}_{d_{l}} \\
\mathbf{0}_{\left(M_{l}-d_{l}\right), d_{l}}
\end{array}\right], \quad \mathbf{U}_{k}=\left[\begin{array}{c}
\mathbf{I}_{d_{k}} \\
\mathbf{0}_{\left(N_{k}-d_{k}\right), d_{k}}
\end{array}\right]
$$

- Channels:

$$
\mathbf{H}_{k l}=\left[\begin{array}{cc}
\mathbf{0}_{d_{k}, d_{l}} & \mathbf{A}_{k l} \\
\mathbf{B}_{k l} & \mathbf{C}_{k l}
\end{array}\right]
$$

## Step 2: Rank checking

To check if the matrix $\boldsymbol{\Psi}$ defining the mapping

$$
\theta:\left(\left\{\dot{\mathbf{U}}_{k}\right\}_{k \in \Phi_{R},},\left\{\dot{\mathbf{V}}_{l}\right\}_{l \in \Phi_{T}}\right) \mapsto\left\{\dot{\mathbf{U}}_{k}^{T} \mathbf{B}_{k l}+\mathbf{A}_{k l} \dot{\mathbf{V}}_{l}\right\}_{(k, l) \in \Phi}
$$

is full row rank.
Example: 3-user channel

| Link | TX1 | TX2 | TX3 | RX1 | RX2 | RX3 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $(1,2)$ | $\Psi_{12}^{(A)}$ | 0 | 0 | 0 | $\Psi_{12}^{(B)}$ | 0 |
| $(1,3)$ | $\Psi_{13}^{(\mathcal{A})}$ | 0 | 0 | 0 | 12 | $\Psi_{13}^{(B)}$ |
| $(2,1)$ | 0 | $\Psi_{21}^{(A)}$ | 0 | $\Psi_{21}^{(B)}$ | 0 | 0 |
| $(2,3)$ | 0 | $\psi_{23}^{(A)}$ | 0 | 0 | 0 | $\Psi_{23}^{(B)}$ |
| $(3,1)$ | 0 | 0 | ${ }_{31}^{(A)}$ | $\Psi_{31}^{(B)}$ | 0 | 0 |
| $(3,2)$ | 0 | 0 | $\Psi_{32}\left(\frac{1}{4}\right.$ | 0 | $\Psi_{32}{ }^{(B)}$ | 0 |

where $\boldsymbol{\Psi}_{k l}^{(A)}=\left(\mathbf{A}_{k l} \otimes \mathbf{I}_{d_{k}}\right) \mathbf{K}_{\left(N_{k}-d_{k}\right), d_{k}}$ and $\boldsymbol{\Psi}_{k l}^{(B)}=\mathbf{I}_{d_{l}} \otimes \mathbf{B}_{k l}^{T}$

## Numerical results

What can this test be used for?

- Check the feasibility of an arbitrary interference channel
- Floating point test is numerically robust for $100+$ antennas/node, e.g., $(86 \times 139,25)^{8}$
- Exact arithmetic test gives a conclusive answer

Feasibility problem belongs to the BPP complexity class

- Extensive evaluation of feasibility in families of systems
- Disprove conjectures by finding counterexamples:

| System | $(11 \times 29, d)^{4}$ | $(19 \times 71, d)^{5}$ | $(29 \times 139, d)^{6}$ |
| :---: | :---: | :---: | :---: |
| Conj. ${ }^{1}$ DoF $\frac{M N}{M+N}$ | 7.975 | 14.989 | 23.994 |
| Actual DoF | 8 | 15 | 24 |

- Powerful tool to obtain research insights, intuitions, establish new conjectures, etc.

[^0]
## A conjecture on the DoF of $(M \times N, d)^{K}$ systems



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## Outline



## Motivation

## Why should we care about the number of solutions?

- Further reveals the mathematical richness of the problem
- Link the IA problem to other, well-studied, combinatorial/graph theory counting problems
- Generalizes the feasibility problem

$$
\text { Infeasible } \Leftrightarrow \text { Number of solutions }=0
$$

- Prediction of large system performance [Schmidt et al., 2010]
- Measure of the algebraic complexity required to compute a solution [Bresler et al., 2014]


## Third contribution: number of solutions, a general formula

 The dimension of the solution set is $s=\operatorname{dim}(\mathcal{V})-\operatorname{dim}(\mathcal{H})$- Negative: 0 solutions
- Positive: 0 or $\infty$ solutions
- Zero: Finite number of solutions

Theorem: If $s=0$, for every choice of $\left\{\boldsymbol{H}_{k l}\right\}$ out of a zero measure set, the IA problem has exactly $S$ alignment solutions given by

$$
S=C f_{H \in\left\|H_{k \|}\right\|_{F=1}} \operatorname{det}\left(\Psi \Psi^{H}\right) d H=C \cdot E\left[\operatorname{det}\left(\Psi \Psi^{H}\right)\right]
$$

How does this generalize our feasibility results?

- Checking feasibility: Pick a canonical IA solution at random and check whether $\operatorname{det}\left(\boldsymbol{\Psi} \boldsymbol{\Psi}^{H}\right) \neq 0$
- Counting solutions: Average $\operatorname{det}\left(\boldsymbol{\Psi} \boldsymbol{\Psi}^{H}\right)$ over all canonical solutions with unit Frobenius norm
González, Santamaría \& Beltrán, TIT, 2nd review round


## Example: the $(2 \times 2,1)^{3}$ system, 2 solutions ${ }^{2}$

- We take uniformly distributed canonical solutions:

$$
\mathbf{H}_{k l}=\left[\begin{array}{cc}
0 & a_{k l} \\
b_{k l} & c_{k l}
\end{array}\right] \longrightarrow \overline{\mathbf{H}}_{k l}=\frac{\mathbf{H}_{k l}}{\left\|\mathbf{H}_{k l}\right\|_{F}},
$$

where $a_{k l}, b_{k l}, c_{k l} \sim C N(0,2)$, i.i.d.

- The $6 \times 6$ matrix $\boldsymbol{\Psi}$ defining the mapping is

[^1]
## Example: the $(2 \times 2,1)^{3}$ system (cont'd)

- The number of solutions is

$$
S=C f_{\boldsymbol{H} \in\left\|\boldsymbol{H}_{k}\right\| \|_{F}} \operatorname{det}\left(\boldsymbol{\Psi} \boldsymbol{\Psi}^{H}\right) d H=C \cdot E\left[|\operatorname{det}(\boldsymbol{\Psi})|^{2}\right]
$$

where $C=3^{6}=729$ for this scenario

- Expanding the determinant of $\boldsymbol{\Psi}$ (squared) along its first column we get

$$
S=3^{6} \cdot 2 \cdot E\left[\left|\frac{b_{12}}{\left\|\mathbf{H}_{12}\right\|_{F}^{2}}\right|^{2}\right]^{6}
$$

where $\left|\frac{b_{12}}{\left\|\boldsymbol{H}_{12}\right\|_{F}^{2}}\right|^{2} \sim \operatorname{Beta}(1,2)$ with mean $1 / 3$

- Consequently,

$$
S=3^{6} \cdot 2 \cdot\left(\frac{1}{3}\right)^{6}=2 \text { solutions }
$$

## Single-beam scenarios: Closed-form solution

Theorem: The number of IA solutions for an arbitrary single-beam scenario with $s=0$ is given by

$$
S=\frac{\operatorname{per}(T)}{\prod_{k}\left(N_{k}-1\right)!\prod_{l}\left(M_{I}-1\right)!}
$$

where $\mathbf{T}$ is the matrix built by replacing the non-zero elements of $\boldsymbol{\Psi}$ by ones and $\operatorname{per}(\mathbf{T})$ denotes its permanent.

- Permanent much harder to compute than determinant
- Permanents of $0 / 1$ matrices appear in many counting problems
- Perfect matchings in bipartite graphs, regular digraphs,...
- Closed-form formulas, e.g.,

$$
(2 \times(K-1), 1)^{K} \text { systems, } S=\text { round }\left(\frac{K!}{e}\right)
$$

- Bounds on the growth rate of the number of solutions


## Multi-beam scenarios: Monte Carlo approximation

Input: Relative error, $\varepsilon$; number of antennas, $\left\{M_{k}\right\}$ and $\left\{N_{k}\right\}$, and streams, $\left\{d_{k}\right\}, \forall k \in \mathcal{K}$
Output: Approximate number of IA solutions, $E_{n}$ begin

```
n\leftarrow1
```

repeat
Generate a set of random matrices $\left\{\mathbf{A}_{k l}\right\},\left\{\mathbf{B}_{k l}\right\}$ and $\left\{\mathbf{C}_{k l}\right\}$ with i.i.d. $\mathcal{C N}(0,2)$ entries
Build canonical channel matrices $\left\{\mathbf{H}_{k l}\right\}$ Normalize every channel matrix $\mathbf{H}_{k l}$ such that $\left\|\mathbf{H}_{k l}\right\|_{F}=1$ Build the matrix $\boldsymbol{\Psi}$
$D_{n} \leftarrow C \operatorname{det}\left(\boldsymbol{\Psi} \boldsymbol{\Psi}^{H}\right)$
Calculate mean, $E_{n}$, and variance, $\sigma_{n}$
$n \leftarrow n+1$
until $\frac{\sigma_{n}}{\sqrt{n} E_{n}}<\varepsilon$

## Some examples

|  | $(2 \times(K-1), 1)^{K}$ | $(3 \times(K-2), 1)^{K}$ <br> Exact $/$ Approx. | $(5 \times(2 K-3), 2)^{K}$ <br> Approx. |
| :---: | :---: | :---: | :---: |
| 3 | Exact / Approx. | $2 / 2 \pm 1.0 \%$ | $1 / 1 \pm 0.5 \%$ |
| 4 | $9 / 9 \pm 1.6 \%$ | $9 / 9 \pm 1.6 \%$ | 1 |
| 5 | $44 / 44 \pm 2.6 \%$ | $216 / 216 \pm 1.5 \%$ | $3700 \pm 0.1 \%$ |
| 6 | $265 / 266 \pm 3.3 \%$ | $7570 / 7291 \pm 5.5 \%$ | $7258239 \pm 17.8 \%$ |

- Number of solutions grows rapidly



## Example: $(3 \times 5,1)^{6}$ system, 7570 solutions

Distinct solutions $\Rightarrow$ Extremely different performance


- Observation: Most of the sum-rate gain is obtained by picking the best out of a small subset of solutions
- Is there a systematic way to compute distinct IA solutions?


## Outline



## Alternating minimization

Research on IA has given rise to a plethora of algorithms, most of them based on alternating minimization
Originally proposed to minimize the interference leakage ${ }^{3}$

$$
\mathrm{IL}=\sum_{k \neq 1}\left\|\mathbf{U}_{k}^{H} \mathbf{H}_{k l} \mathbf{V}\right\|_{l} \|_{F}^{2}
$$

- Typically slow (linear convergence rate)
- Bounces and circles around minima
- Monotone convergence
- No guaranteed convergence
- No systematic way of getting L different solutions


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[^7]
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[^8]
## Homotopy continuation

Basic idea: define a parametrized transformation or homotopy that gradually deforms a trivially solvable system, or start system, into the target system that we want to solve

A simple homotopy: a series of MIMO channels obtained as a convex combination of a start channel, $\overline{\mathbf{H}}_{k l}$, and the target channel, $\mathbf{H}_{k l}$

$$
\mathbf{G}_{k l}\left(\mathbf{U}_{k}^{H}, \mathbf{V}_{l}, t\right):=\mathbf{U}_{k}^{H} \underbrace{\left((1-t) \overline{\mathbf{H}}_{k l}+t \mathbf{H}_{k l}\right)}_{\mathbf{H}_{k l}(t)} \mathbf{V}_{l}, \quad \forall k, l \in \Phi \text { and } t \in[0,1]
$$

The combination is controlled by the continuation parameter, $t$

González \& Santamaría, ICASSP 2011; González, Fanjul \& Santamaría, ICASSP 2014

## Homotopy Continuation



## Start system: the inverse IA problem

How do we find an appropriate easy-to-solve system?

- Consider the inverse IA problem:

$$
\begin{aligned}
& \text { Given } \mathbf{U}_{k}^{H} \text { and } \mathbf{V}_{l} \text {, find } \overline{\mathbf{H}}_{k l} \text { such that } \\
& \qquad \mathbf{U}_{k}^{H} \overline{\mathbf{H}}_{k l} \mathbf{V}_{l}=\mathbf{0}, \quad \forall k, l \in \Phi
\end{aligned}
$$

- A linear equation per each $\overline{\mathbf{H}}_{k l}$ (total of $K(K-1)$ equations which are solved independently)
- Every solution can be parametrized as

$$
\overline{\mathbf{H}}_{k l}=\mathbf{X}_{k l}-\mathbf{A}_{k} \mathbf{A}_{k}^{H} \mathbf{x}_{k l} \mathbf{B}_{l} \mathbf{B}_{l}^{H}
$$

where $\mathbf{A}_{k}$ and $\mathbf{B}_{l}$ are orthonormal bases of $\mathbf{U}_{k}$ and $\mathbf{V}_{l}$, respectively, and $\mathbf{X}_{k l}$ is a non-zero arbitrary matrix.

## Path-following procedure

A first order approximation of the homotopy function

$$
\begin{array}{r}
\mathbf{G}_{k l}\left(\mathbf{U}_{k}^{H}+\Delta \mathbf{U}_{k}^{H}, \mathbf{V}_{l}+\Delta \mathbf{V}_{l}, t+\Delta t\right)= \\
\mathbf{U}_{k}^{H} \mathbf{H}_{k l}(t) \mathbf{V}_{l}+ \\
\Delta \mathbf{U}_{k}^{H} \mathbf{H}_{k l}(t) \mathbf{V}_{l}+\mathbf{U}_{k}^{H} \mathbf{H}_{k l}(t) \Delta \mathbf{V}_{l}+ \\
\mathbf{U}_{k}^{H}\left(\mathbf{H}_{k l}-\overline{\mathbf{H}}_{k l}\right) \mathbf{V}_{l} \Delta t \quad \forall k, l \in \Phi
\end{array}
$$

gives rise to a two-step path-following procedure:

1. Euler prediction
2. Newton correction

## Step 1: Euler prediction



If the current point is in the path, we want the predicted solution at $t+\Delta t$ to be as close to the path as possible:

$$
\mathbf{G}_{k l}\left(\mathbf{U}_{k}^{H}+\Delta \mathbf{U}_{k}^{H}, \mathbf{V}_{l}+\Delta \mathbf{V}_{l}, t+\Delta t\right) \approx \mathbf{0}
$$

- Precoder and decoder updates, $\Delta \mathbf{V}$, and $\Delta \mathbf{U}_{k}^{H}$, are obtained by solving the system of linear equations:

$$
\begin{aligned}
& \Delta \mathbf{U}_{k}^{H} \mathbf{H}_{k l}(t) \mathbf{V}_{l}+\mathbf{U}_{k}^{H} \mathbf{H}_{k l}(t) \Delta \mathbf{V}_{l}= \\
& -\mathbf{U}_{k}^{H}\left(\mathbf{H}_{k l}-\overline{\mathbf{H}}_{k l}\right) \mathbf{V}, \Delta t \quad \forall k, l \in \Phi
\end{aligned}
$$

## Step 2: Newton correction


$\Delta t=0$
If the current point $\left(\left\{\mathbf{U}_{k}\right\},\left\{\mathbf{V}_{l}\right\}, t\right)$ is not as close to the path as we would like, i.e. the entries of $\mathbf{G}_{k l}\left(\mathbf{U}_{k}^{H}, \mathbf{V}_{l}, t\right)$ are larger than a predefined tolerance, we can hold $t$ constant by setting $\Delta t=0$ and obtain the Newton correction step.

- Again, precoder and decoder updates, $\Delta \mathbf{V}_{l}$ and $\Delta \mathbf{U}_{k}^{H}$ are obtained by solving a system of linear equations:

$$
\Delta \mathbf{U}_{k}^{H} \mathbf{H}_{k l}(t) \mathbf{V}_{l}+\mathbf{U}_{k}^{H} \mathbf{H}_{k l}(t) \Delta \mathbf{V}_{l}=-\mathbf{U}_{k}^{H} \mathbf{H}_{k l}(t) \mathbf{V}_{l}, \forall k, l \in \Phi
$$

## Features

- Quadratic convergence rate
- Faster than previously known algorithms in tight systems
- Systematic way to compute $L$ distinct solutions:
- $L$ trivial system solutions $\Rightarrow L$ target system solutions
- Possibility to use "pre-computed solutions"
- Simple extension to other networks (X networks, structured channels, etc.)
- Rank conditions explicitly enforced by adding $\mathbf{U}_{k}^{H} \mathbf{H}_{k k} \mathbf{V}_{k}=\mathbf{I}$ as an additional equation (involves a change of basis at RX)


## Gauss-Newton algorithm

Observation: for ICs, a sequence Newton step converges globally Explanation:

- Newton step for system solving can be regarded as a Gauss-Newton method for IL minimization
- In GN the cost function is approximated by a convex function


## Interference leakage convexifies as it approaches zero

- We can distinguish two operational regimes:

1. Approximation $(I L \geq \mu)$ : non-monotone convergence
2. Exact $(I L<\mu)$ : quadratic convergence

## Comparison GN vs HC:

- Faster convergence at the expense of the capacity to track different solutions

González, Lameiro \& Santamaría, SPL, 2014

## Convergence speed: $(5 \times 5,2)^{4}$ system, 3700 solutions



## Sum-rate performance: $(3 \times 3,1)^{5}$ system, 216 solutions



Incremental SNR algorithm by Schmidt et al., 2013 (best-performing algorithm in single-beam networks)

## Conclusions and further work

- Closed-form feasibility conditions for single-beam systems
- Numerical feasibility test for general scenarios

Closed-form number of solutions for single-beam systems

- Monte Carlo approx. of no. of sols. in general scenarios
- Gauss-Newton and homotopy continuation algorithms
- Closed-form results for multi-beam networks (derive DoF bounds by network flow analysis, feasibility results from structure of $\boldsymbol{\Psi}, \ldots$ )
- Combinatorial interpretation of the number of solutions in multi-beam networks
- Algorithms on Riemannian manifolds, e.g. Grassmann, Stiefel
- Distributed versions of both HC and GN algorithms

Extensions to rank-deficient or structured channels, asymmetric complex-signaling,...

## Publications derived from this thesis

J1 Ó. González, C. Beltrán, and I. Santamaría, "A Feasibility Test for Linear Interference Alignment in MIMO Channels with Constant Coefficients", IEEE Transactions on Information Theory, vol. 60, no. 3, pp. 1840-1856, Mar. 2014.
J2 Ó. González, C. Lameiro, and I. Santamaría, "A Quadratically Convergent Method for Interference Alignment in MIMO Interference Channels", IEEE Signal Processing Letters, vol. 21, no. 11, pp. 1423-1427, Nov. 2014.
J3 Ó. González, C. Beltrán, and I. Santamaría, "On the Number of Interference Alignment Solutions for the K-User MIMO Channel with Constant Coefficients", submitted to IEEE Transactions on Information Theory (2nd review round), Jan. 2013 arXiv: 1301.6196.
J4 J. Fanjul, Ó. González, and I. Santamaría, "Homotopy continuation algorithms for interference alignment in arbitrary networks", in preparation.

C1 Ó. González and I. Santamaría, "Interference Alignment in Single-Beam MIMO Networks Via Homotopy Continuation", in IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP), Prague, Czech Republic, May 2011, pp. 3344-3347.
C2 Ó. González, I. Santamaría, and C. Beltrán, "A General Test to Check the Feasibility of Linear Interference Alignment", in 2012 IEEE International Symposium on Information Theory Proceedings (ISIT), Cambridge, MA, USA, Jul. 2012, pp. 2481-2485.
C3 Ó. González, C. Lameiro, J. Vía, C. Beltrán, and I. Santamaría, "Computing the Degrees of Freedom for Arbitrary MIMO Interference Channels", in 2013 IEEE International Conference on Acoustics, Speech, and Signal Processing (ICASSP), Vancouver, Canada, May 2013, pp. 4399-4403.
C4 Ó. González, I. Santamaría, and C. Beltrán, "Finding the Number of Feasible Solutions for Linear Interference Alignment Problems", in 2013 IEEE International Symposium on Information Theory (ISIT), Istanbul, Turkey, Jul. 2013, pp. 384-388.
C5 Ó. González, J. Fanjul, and I. Santamaría, "Homotopy Continuation for Vector Space Interference Alignment in MIMO X Networks", in 2014 IEEE International Conference on Acoustic, Speech and Signal Processing (ICASSP), Florence, Italy, May 2014, pp. 6232-6236.

## Other production and impact indicators

- Interference Alignment Toolbox (IAbox) http://github.com/masdeseiscaracteres/IAbox
- Online feasibility test (1500 executions, 45 locations, 17 countries) http://gtas.unican.es/IAtest


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- Related publications:
- IA: experimental work, transceiver design,...
- MIMO systems
- Physical layer security
- Total production:
- 6 journal papers (3 published, 2 under review, 1 in preparation)
- 15 conference papers


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- IA: experimental work, transceiver design,...
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Thank you for your attention

Network capacity - the holy grail of information theory


- Capacity: set of all possible rates supported by a network
- Studied for decades but still open (even for the 2-user IC)
- Several approximations attempted: Degrees of freedom (DoF)
- Pre-log factor of the capacity
- High-SNR slope of sum-rate
- Number of non-interfering signal dimensions
Our main focus: Interference alignment (IA)
- Achievability of $1 / 2 \mathrm{DoF} /($ user and signaling dimension) in an interference channel [Cadambe \& Jafar, 2008]
- IA has been shown to be DoF-optimal in many other scenarios


## Outline



## Can we do better? - A network flow approach



Supply-Demand Theorem (Gale, 1957; Mirsky, 1968): A feasible flow exists if and only if

$$
\sum_{k \in B} b_{k}-\sum_{l \in \bar{A}} a_{l} \leq \sum_{\substack{k \in B \\ l \in A}} c_{k l}, \quad \forall A, B \subseteq \mathcal{K}
$$

Identifying the properness conditions with the S-D Theorem:

$$
\sum_{I:(k, l) \in \phi}\left(M_{l}-d_{l}\right) d_{l}+\sum_{k:(k, l) \in \phi}\left(N_{k}-d_{k}\right) d_{k} \geq \sum_{(k, l) \in \phi} d_{l} d_{k} \forall \phi \subseteq \Phi
$$

- Supplies: $a_{l}=\left(M_{l}-d_{l}\right) d_{l}$
- Demands: $b_{k}=d_{k} \max \left(\sum_{l} d_{l}-N_{k}, 0\right)$
- Capacity: $c_{k l}=d_{k} d_{l} \forall(k, l) \in \Phi, c_{k l}=0$ otherwise

Theorem: If the maximum flow, $F$, in the transport network does not fulfill the aggregate demand, i.e.,

$$
F<\sum_{k} d_{k} \max \left(\sum_{l} d_{l}-N_{k}, 0\right)
$$

Then, no other feasible flow exists and the system is not feasible.

Identifying the properness conditions with the S-D Theorem:

$$
\sum_{l:(k, l) \in \phi}\left(M_{l}-d_{l}\right) d_{l}+\sum_{k:(k, l) \in \phi}\left(N_{k}-d_{k}\right) d_{k} \geq \sum_{(k, l) \in \phi} d_{l} d_{k} \forall \phi \subseteq \Phi
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Theorem: If the maximum flow, $F$, in the transport network does not fulfill the aggregate demand, i.e.,

Goldberg's maximum flow algorithm $O\left(K^{3}\right)$
Then, no other feasible flow exists and the system is not feasible.

## Example 1: Demand not fulfilled $\Rightarrow$ Infeasible system



Figure: Maximum flow for the $(4 \times 2,1)(2 \times 2,1)^{2}(2 \times 4,1)$ system

Remark: The opposite does not hold, see next example

## Example 2: Demand fulfilled $\nRightarrow$ Feasible system



Figure: Maximum flow for the $(4 \times 4,1)(2 \times 2,1)^{3}$ system

A feasible flow fulfilling the demands does not mean the system is feasible

OK, but is it feasible or not?

## Single-beam systems

- Properness is a necessary and sufficient condition for the feasibility of single-beam systems

1. Maximum flow algorithms provide a conclusive answer in polynomial time
2. A closed-form solution is also possible

Theorem: Consider a fully connected IC where the users are sorted such that $M_{k} \geq M_{k+1}$ and $N_{k} \leq N_{k+1}$ if $M_{k}=M_{k+1}$. Then, interference alignment in this network is feasible if and only if

$$
\sum_{i=1}^{k} \max \left(K-N_{i}, 0\right)^{* *} \geq \sum_{i=1}^{k}\left(M_{i}-1\right) \forall k \in \mathcal{K}
$$

where ${ }^{* *}$ denotes the I-restricted conjugate partition.

## Single-beam systems

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Linear time-complexity: $O(K)$
where ${ }^{* *}$ denotes the I-restricted conjugate partition.

## Example 1: Infeasible system $(4 \times 2,1)(2 \times 2,1)^{2}(2 \times 4,1)$

- TX/RX antennas, supplies and demands

$$
\begin{array}{ll}
M=(4,2,2,2) & a=\left(M_{i}-1\right)=(3,1,1,1) \\
N=(2,2,2,4) & b=\left(K-N_{i}\right)=(2,2,2,0)
\end{array}
$$

- I-restricted conjugate partition of $b$ :

$$
b^{* *}=(2,2,2,0)
$$



- Does $b^{* *}$ majorize $a$ ?

$$
(2,4,6,6) \nsupseteq(3,4,5,6) \quad \Rightarrow \quad \text { Infeasible }
$$

## Example 2: Feasible system $(4 \times 4,1)(2 \times 2,1)^{3}$

- TX/RX antennas, supplies and demands

$$
\begin{array}{ll}
M=(4,2,2,2) & a=\left(M_{i}-1\right)=(3,1,1,1) \\
N=(4,2,2,2) & b=\left(K-N_{i}\right)=(0,2,2,2)
\end{array}
$$

- I-restricted conjugate partition of $b$ :

$$
b^{* *}=(3,2,1,0)
$$



- Does $b^{* *}$ majorize $a$ ?

$$
(3,5,6,6) \geq(3,4,5,6) \quad \Rightarrow \quad \text { Feasible }
$$

The DoF of $(M \times N, d)^{3}$ systems


A conjecture on the DoF of $(M \times N, d)^{K}$ systems
$d / N$ Piecewise

A conjecture on the DoF of $(M \times N, d)^{K}$ systems Two regimes: below and above the threshold

$$
\lambda=1 / 2\left(K-1-\sqrt{(K-1)^{2}-4}\right):
$$

1. Piecewise linear regime (proved by Liu and Yang, 2013):

$$
\begin{gathered}
d^{\star}=\left\{\begin{array}{ll}
\frac{\gamma(p)+1}{\gamma(p)(K+1)} M, & \gamma^{\prime}(p) \leq \frac{M}{N} \leq \gamma(p) \\
\frac{\gamma(p)+1}{K+1} N, & \gamma(p) \leq \frac{M}{N} \leq \gamma^{\prime}(p+1)
\end{array} \quad p \in \mathbb{Z}^{+} .\right. \\
\gamma(p)=\frac{\sum_{k=-(p-1)}^{(p-1)} \lambda^{k}}{\sum_{k=-p}^{p} \lambda^{k}} \text { and } \gamma^{\prime}(p)=\lambda \frac{\sum_{k=0}^{p-2} \lambda^{2 k}}{\sum_{k=0}^{p-1} \lambda^{2 k}}
\end{gathered}
$$

2. Properness-limited regime (remains unproven):

$$
d^{\star}=\frac{M+N}{K+1}
$$


[^0]:    ${ }^{1}$ Wang, Sun, Jafar, ISIT, 2012

[^1]:    ${ }^{2}$ Cadambe \& Jafar, 2008

[^2]:    ${ }^{3}$ Gomadam et al., 2011

[^3]:    ${ }^{3}$ Gomadam et al., 2011

[^4]:    ${ }^{3}$ Gomadam et al., 2011

[^5]:    ${ }^{3}$ Gomadam et al., 2011

[^6]:    ${ }^{3}$ Gomadam et al., 2011

[^7]:    ${ }^{3}$ Gomadam et al., 2011

[^8]:    ${ }^{3}$ Gomadam et al., 2011

