Homotopy Continuation for Vector Space Interference Alignment in MIMO X Networks

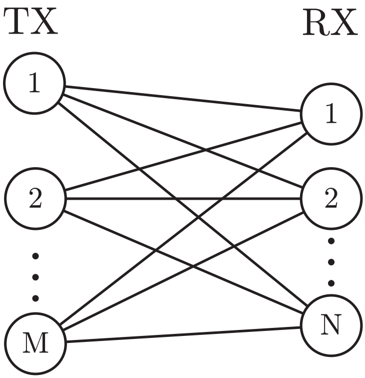
Óscar González, Jacobo Fanjul, and Ignacio Santamaría

University of Cantabria, Spain

ICASSP 2014 - Florence, Italy
May 9, 2014

Contents

X Network: a very general network topology



- Arbitrary number of transmitters and receivers

  Here, both are equal to K
- Arbitrary message demands

  Here, we will assume each transmitter sends d independent data streams to each receiver
- Messages intended for a single receiver (unicast traffic)
- Generalizes IC, IBC, IMAC, XC,...

X Network: problem formulation

Assumption: Generic and unstructured MIMO channels (random and w/o channel extensions)

We will use linear precoding/decoding to cancel out the interference:

$$\mathbf{U}_k^H \mathbf{H}_{kl} \mathbf{V}_{\overline{kl}} = \mathbf{0}, \quad \forall k, l \qquad K^3 (K-1) d^2 \text{ bilinear eqs.}$$

$$\operatorname{rank}\left(\mathbf{U}_k^H \left[\mathbf{H}_{k1}\mathbf{V}_{k1}, \ldots, \mathbf{H}_{kK}\mathbf{V}_{kK}\right]\right) = Kd, \quad \forall k \qquad K \text{ rank constraints}$$

where $\mathbf{V}_{\overline{kl}}$ is defined as the horizontal concatenation of all $\mathbf{V}_{jl}$ such that $j \neq k$, i.e.,

$$\mathbf{V}_{\overline{kl}} \stackrel{\text{def}}{=} \operatorname*{cat}_{j \neq k}\left(\mathbf{V}_{jl}\right).$$

X Network: problem formulation

Assumption: Generic and unstructured MIMO channels (random and w/o channel extensions)

We will use linear precoding/decoding to cancel out the interference:

$$\mathbf{U}_k^H \mathbf{H}_{kl} \mathbf{V}_{\overline{kl}} = \mathbf{0}, \quad \forall k, l$$

$K^3 (K-1) d^2$ bilinear eqs.

where $\mathbf{V}_{\overline{kl}}$ is defined as the horizontal concatenation of all $\mathbf{V}_{jl}$ such that $j \neq k$, i.e.,

$$\mathbf{V}_{\overline{kl}} \stackrel{\text{def}}{=} \underset{j \neq k}{\text{cat}} \left(\mathbf{V}_{jl} \right).$$

Interference Channel: a *nice* particularization

- Equal number of transmitters and receivers (here, K)
- One-to-one message demands
  - Here, each transmitter sends d independent data streams to its corresponding receiver

$$\mathbf{U}_k^H \mathbf{H}_{kl} \mathbf{V}_{ll} = \mathbf{0}, \quad \forall k, \ \forall l \neq k \qquad K(K-1)d^2 \text{ bilinear eqs.}$$

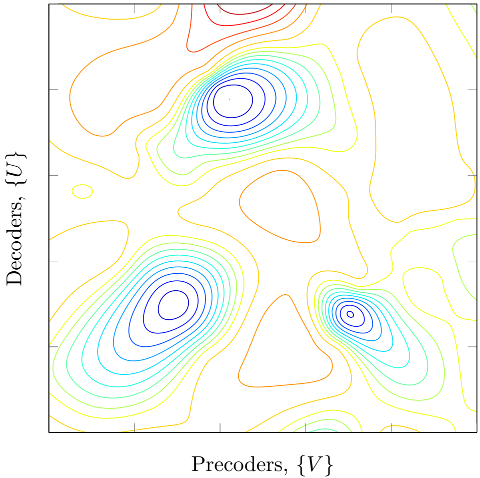
$$\operatorname{rank}\left(\mathbf{U}_k^H \mathbf{H}_{kk} \mathbf{V}_{kk}\right) = d, \quad \forall k \qquad K \text{ rank conditions}$$

Why is the interference channel that *nice*?

- Rank conditions are satisfied automatically if $\mathbf{U}_k$ and $\mathbf{V}_l$ are full column rank, e.g. $\mathbf{U}_k^H \mathbf{U}_k = \mathbf{I}$ and $\mathbf{V}_l^H \mathbf{V}_l = \mathbf{I}$
- This property is exploited by the alternating minimization algorithm [Gomadam et al. 2011]

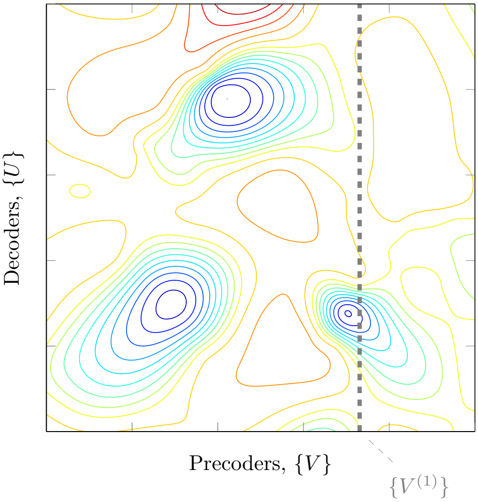
Alternating Minimization

Cost function: $IL = \sum_{k \neq l} ||\mathbf{U}_k^H \mathbf{H}_{kl} \mathbf{V}_{ll}||^2$



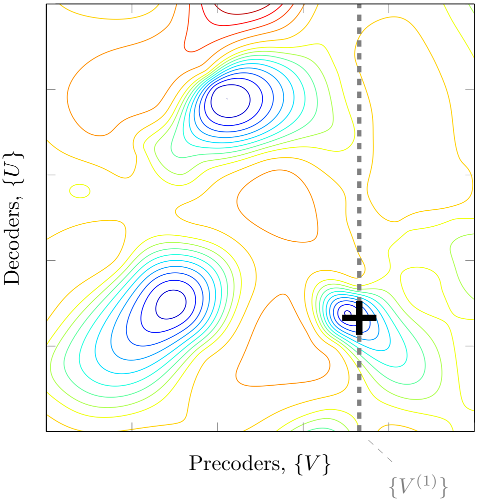
Alternating Minimization

Cost function: $IL = \sum_{k \neq l} ||\mathbf{U}_k^H \mathbf{H}_{kl} \mathbf{V}_{ll}||^2$



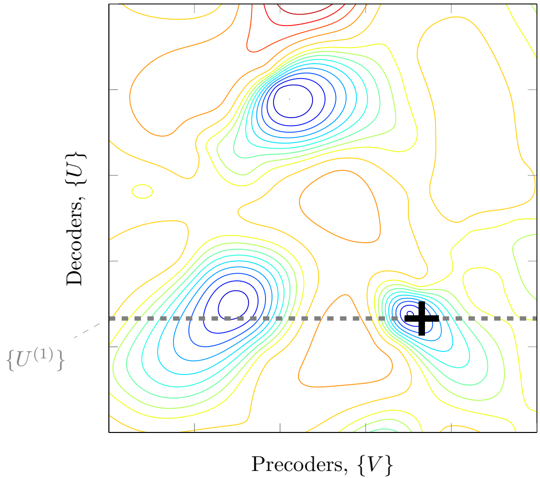
Alternating Minimization

Cost function: $IL = \sum_{k \neq l} ||\mathbf{U}_k^H \mathbf{H}_{kl} \mathbf{V}_{ll}||^2$



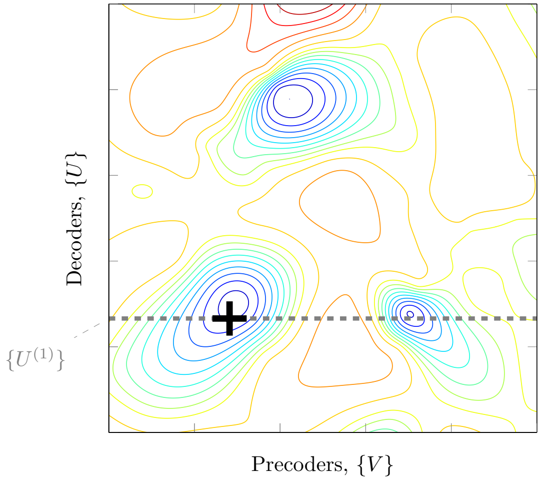
Alternating Minimization

Cost function: $IL = \sum_{k \neq l} ||\mathbf{U}_k^H \mathbf{H}_{kl} \mathbf{V}_{ll}||^2$



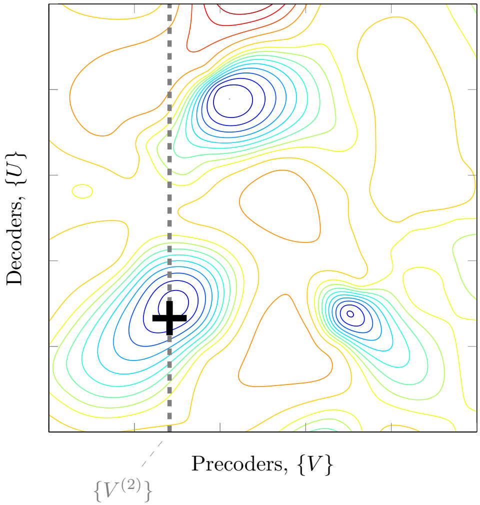
Alternating Minimization

Cost function: $IL = \sum_{k \neq l} ||\mathbf{U}_k^H \mathbf{H}_{kl} \mathbf{V}_{ll}||^2$



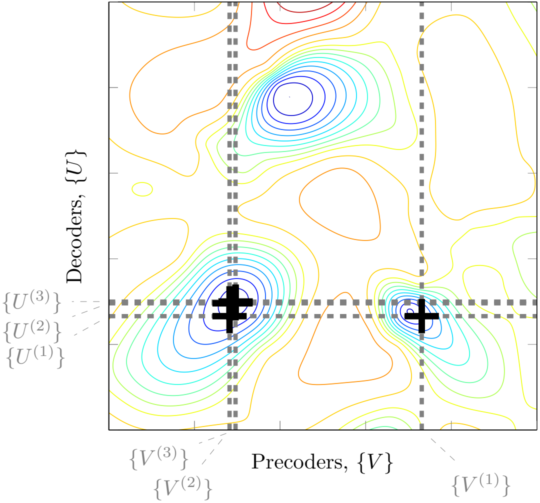
Alternating Minimization

Cost function: $IL = \sum_{k \neq l} ||\mathbf{U}_k^H \mathbf{H}_{kl} \mathbf{V}_{ll}||^2$



Alternating Minimization

Cost function: $IL = \sum_{k \neq l} ||\mathbf{U}_k^H \mathbf{H}_{kl} \mathbf{V}_{ll}||^2$



Alternating Minimization: properties and characteristics

- Simple interpretation and implementation
- Slow convergence (bounces and circles around minima)
- Convergence to a global minimum is not guaranteed

For an interference channel...

- There is a strong numerical evidence that it always converges to the global optimum, i.e. $IL = 0$
- Provides useful insights in the study of feasibility

For an X network...

- It does not guarantee the rank condition (strong coupling: every link acts as both desired and interfering)

What to do instead?

Homotopy Continuation

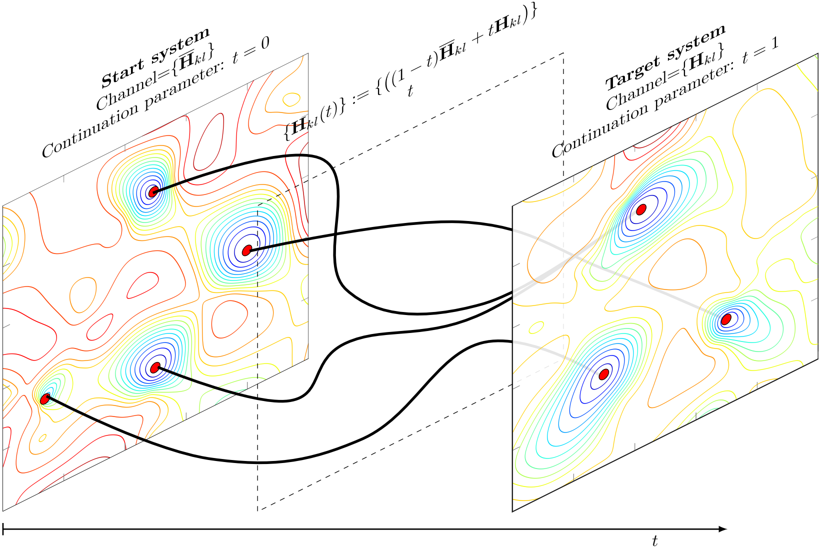
Basic idea: define a parametrized transformation or **homotopy** that gradually deforms a trivially solvable system, or **start system**, into the **target system** that we want to solve

A simple homotopy: a path of MIMO channels obtained as a convex combination of a *start* channel, $\overline{\mathbf{H}}_{kl}$, and the target channel, $\mathbf{H}_{kl}$

$$\mathbf{G}_{kl}(\mathbf{U}_k^H, \mathbf{V}_{\overline{kl}}, t) := \mathbf{U}_k^H \underbrace{\left((1-t)\overline{\mathbf{H}}_{kl} + t\mathbf{H}_{kl}\right)}_{\mathbf{H}_{kl}(t)} \mathbf{V}_{\overline{kl}}, \quad \forall k, l \text{ and } t \in [0,1]$$

The combination is controlled by the continuation parameter, t

Homotopy Continuation



Start system: the *inverse* IA problem

How do we find an appropriate easy-to-solve system?

- Consider the **inverse IA problem**:

  Given $\mathbf{U}_k^H$ *and* $\mathbf{V}_{\bar{k}l}$, *find* $\overline{\mathbf{H}}_{kl}$ *such that*

$$\mathbf{U}_k^H \overline{\mathbf{H}}_{kl} \mathbf{V}_{\bar{k}l} = \mathbf{0}, \quad \forall k, l$$

- A linear equation per each $\overline{\mathbf{H}}_{kl}$ (total of K^2 equations which are solved independently)
- Every solution can be parametrized as

$$\overline{\mathbf{H}}_{kl} = \mathbf{X}_{kl} - \mathbf{F}_k \mathbf{F}_k^H \mathbf{X}_{kl} \mathbf{G}_{\bar{k}l} \mathbf{G}_{\bar{k}l}^H$$

  where $\mathbf{F}_k$ and $\mathbf{G}_{\bar{k}l}$ are orthonormal bases of $\mathbf{U}_k$ and $\mathbf{V}_{\bar{k}l}$, respectively, and $\mathbf{X}_{kl}$ is a non-zero arbitrary matrix.

Path-following procedure

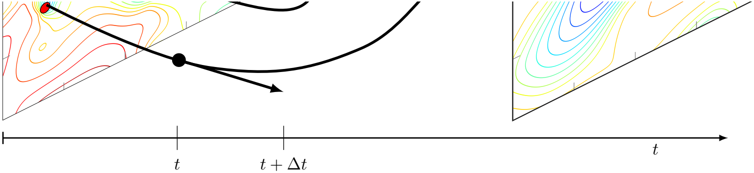
A first order approximation of the homotopy function

$$\begin{aligned}\mathbf{G}_{kl}(\mathbf{U}_k^H + \Delta\mathbf{U}_k^H, \mathbf{V}_{\overline{k}l} + \Delta\mathbf{V}_{\overline{k}l}, t + \Delta t) = \\ \mathbf{U}_k^H \mathbf{H}_{kl}(t)\mathbf{V}_{\overline{k}l} + \\ \Delta\mathbf{U}_k^H \mathbf{H}_{kl}(t)\mathbf{V}_{\overline{k}l} + \mathbf{U}_k^H \mathbf{H}_{kl}(t)\Delta\mathbf{V}_{\overline{k}l} + \\ \mathbf{U}_k^H (\mathbf{H}_{kl} - \overline{\mathbf{H}}_{kl})\mathbf{V}_{\overline{k}l}\Delta t \quad \forall k, l,\end{aligned}$$

gives rise to a two-step path-following procedure:

1. Euler prediction
2. Newton correction

Step 1: Euler prediction

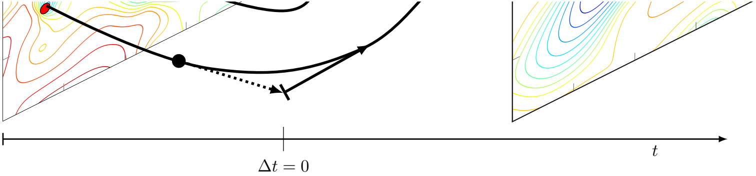


A predicted solution at $t + \Delta t$ has to be as close to the path as possible: $\mathbf{G}_{kl}(\mathbf{U}_k^H + \Delta\mathbf{U}_k^H, \mathbf{V}_{\bar{k}l} + \Delta\mathbf{V}_{\bar{k}l}, t + \Delta t) = \mathbf{0}$
Therefore,

$$\Delta\mathbf{U}_k^H\mathbf{H}_{kl}(t)\mathbf{V}_{\bar{k}l} + \mathbf{U}_k^H\mathbf{H}_{kl}(t)\Delta\mathbf{V}_{\bar{k}l} = -\mathbf{U}_k^H(\mathbf{H}_{kl} - \overline{\mathbf{H}}_{kl})\mathbf{V}_{\bar{k}l}\Delta t \quad \forall k, l.$$

- $\Delta\mathbf{V}_{\bar{k}l}$ and $\Delta\mathbf{U}_k^H$ are obtained by solving a **system of linear equations**

Step 2: Newton correction



If the current point $(\{\mathbf{U}_k\}, \{\mathbf{V}_{jl}\}, t)$ is not as close to the path as we would like, i.e. the entries of $\mathbf{G}_{kl}(\mathbf{U}_k^H, \mathbf{V}_{\bar{kl}}, t)$ are larger than a predefined tolerance, we can hold t constant by setting $\Delta t = 0$ and obtain the Newton correction step:

$$\Delta \mathbf{U}_k^H \mathbf{H}_{kl}(t) \mathbf{V}_{\bar{kl}} + \mathbf{U}_k^H \mathbf{H}_{kl}(t) \Delta \mathbf{V}_{\bar{kl}} = -\mathbf{U}_k^H \mathbf{H}_{kl}(t) \mathbf{V}_{\bar{kl}}, \ \forall k, l$$

- Again, precoder and decoder updates, $\Delta \mathbf{V}_{\bar{kl}}$ and $\Delta \mathbf{U}_k^H$ are obtained by solving a **system of linear equations**

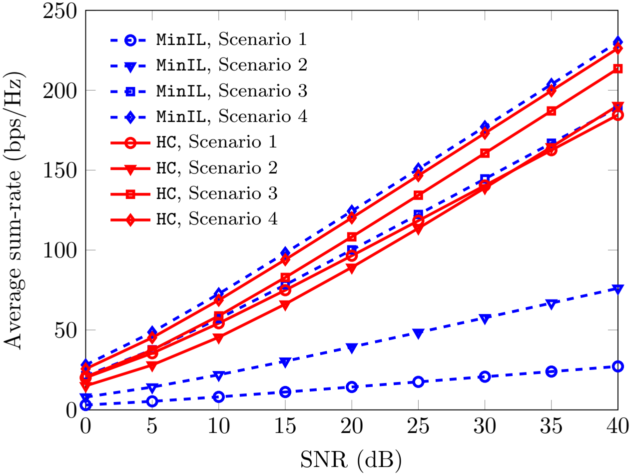
DoF performance

- $K = 4$ users (K^2 desired links)
- $d = 1$ stream/link ($K^2 d^2 = 16$ simultaneous streams)
- Four scenarios (1-4, from tightest to loosest according to the bound $d \leq \frac{n_T + n_R}{K^2+1}$ [Sun et al. 2012])

Scenario	1	2	3	4
$n_T \times n_R$	8×9	9×9	10×10	11×11
`MinIL`	1.00	1.00	0.56	0.01
`HC`	0.18	0.15	0.02	0.00

Prob[DoF $< K^2 d = 16$] or probability of not achieving the requested number of degrees of freedom

Sum-rate performance



Conclusion

Summary

- New algorithm to compute interference alignment solutions
- Provides superior performance in terms of achieved DoF/sum-rate
- Useful as a numerical means to provide evidence of feasibility

Future work

- Explicitly include rank constraints
- Extension to structured channels (diagonal, block diagonal,. . .) which appear when channel extensions/ asymmetric complex signalling are used

Thank you for your attention