

Approximate closed-form expression for the ergodic capacity of MISO and SIMO systems

J. Pérez, J. Ibáñez, L. Vielva, I. Santamaría
Communications Engineering Department
University of Cantabria
Santander, Spain
jperez@gtas.dicom.unican.es

Abstract— In this paper we derive a simple and tight close-form approximation for the ergodic capacity of SIMO, MISO, and orthogonalized MIMO channels. The expression is valid for any SNR range, any number of antennas and for a wide variety of fading channels including correlated and unbalanced Rayleigh and Ricean channels.

Keywords: Channel capacity, multi-antenna system, radio channel model

I. INTRODUCTION

The performance of multi-antenna systems strongly depends on the specific channel characteristics. Usually, the bit error rate, and the ergodic and outage capacity are employed as performance metrics. In this work we focus on the ergodic capacity of multi-antenna systems in narrowband fading channels.

Analytic bounds and approximations for the ergodic capacity of multi-antenna systems can be found in the technical literature, assuming Rayleigh or Ricean fading channels [1] – [8]. In this work we use a physical channel model to derive a simple and tight closed-form approximation for the ergodic capacity of SIMO, MISO and orthogonalized MIMO channels. Orthogonalized MIMO (O-MIMO) channels have practical interest because they arise when space-time block coding is used. The derived expression implicitly takes into account the specific multi-path structure of the channel, and encompasses the Rayleigh and Ricean fading channels as particular cases. Section II describes the channel model that we will use in the analysis. In section III we derive the analytical expression for the ergodic capacity as a function of a number of channel parameters of the channel model. In section IV we present some simulation results that show the tightness of the approximation in different propagation conditions and channel configurations.

II. CHANNEL MODEL

For a general analysis we consider a MIMO channel model with n_R receive antennas and n_T transmit antennas. The MISO and SIMO cases will be particular cases with $n_R = 1$ and $n_T = 1$, respectively. The channel is assumed to be flat over the frequency band of interest. The channel is modeled using the so-called ISLAC (Independent Stochastic Local Area Channel) model [9]. Then, the base-band equivalent MIMO

channel matrix can be expressed as the sum of the contributions of M matrices corresponding to M uniform plane waves due to the multi-path propagation:

$$\mathbf{H} = \sum_{m=1}^M \mathbf{V}_m \exp(j\mathbf{f}_m), \quad (1)$$

Where, in general, \mathbf{H} and the \mathbf{V}_m are $n_R \times n_T$ matrices. In the SIMO and MISO cases, \mathbf{H} and \mathbf{V}_m will be column and row vectors with n_R and n_T entries, respectively. According to the ISLAC model, the multi-path phases \mathbf{f}_m are independent uniformly distributed random variables, each one associated with a multi-path wave. The matrices \mathbf{V}_m are the parameters of the channel model. They are deterministic matrices that depend on the array configurations, antennas characteristics, and on the specific propagation scenario. The values of the \mathbf{V}_m matrices determine the fading distributions as well as the correlations of the MIMO channel matrix. Considering the statistical distribution of the multi-path phases \mathbf{f}_m , the covariance matrix of the channel can be expressed analytically as a function of the matrices \mathbf{V}_m as follows

$$\mathbf{R} = E[\text{vec}(\mathbf{H}) \text{vec}(\mathbf{H})^H] = \sum_{m=1}^M \text{vec}(\mathbf{V}_m) \text{vec}(\mathbf{V}_m)^H. \quad (2)$$

This channel model allow us to model a wide variety of MIMO channels including the correlated Rayleigh and Ricean fading channels as particular cases [9]. For example, it can be shown that the channel behaves as a balanced uncorrelated Rayleigh channel when the number of multi-path waves (M) is high and the amplitudes of the entries of the matrices \mathbf{V}_m have equal amplitude. The above channel plus LOS contribution \mathbf{V}_0 will model a Ricean fading channel. For example, to model a correlated Rayleigh channel with specific transmit and receive covariance matrices (\mathbf{R}_T and \mathbf{R}_R) we can use the same matrices as in the uncorrelated case, but multiplied by the covariance matrices

$$\mathbf{H} = \sum_{m=1}^M \mathbf{R}_R^{1/2} \mathbf{V}_m \mathbf{R}_T^{1/2} \exp(j\mathbf{f}_m). \quad (3)$$

III. ERGODIC CAPACITY

We assume that the channel is known at the receiver. For a given realization of the channel, the capacity (in bps/Hz) of a SIMO, MISO or O-MIMO channel can be written in a general expression as follows [10]

$$C = \mathbf{h} \log_2(D), \quad D = 1 + \frac{\mathbf{r}}{\mathbf{b} \mathbf{a}} \|\mathbf{H}\|_F^2, \quad (4)$$

where \mathbf{r} is the average signal-to-noise ratio at the receiver antennas, $\|\mathbf{H}\|_F^2$ is the squared Frobenius norm of the channel matrix and \mathbf{a} is the average path-loss, which is given by

$$\mathbf{a} = \frac{E[\|\mathbf{H}\|_F^2]}{n_R n_T}, \quad (5)$$

being $E[\cdot]$ the expectation operator. The parameters \mathbf{h} and \mathbf{b} in (4) take different values depending on the system configuration (SIMO, MISO or O-MIMO) and on the channel knowledge at the transmitter. \mathbf{b} equals 1 in SIMO and in MISO configurations when the channel is known at the transmitter, and \mathbf{b} equals n_T in MISO or O-MIMO configurations without channel knowledge at the transmitter. The parameter \mathbf{h} equals 1 in SIMO and MISO configurations, and it equals the code rate of the space-time block coding in the O-MIMO case. Therefore, considering the above values of \mathbf{h} and \mathbf{b} , (4) is valid for SIMO, MISO and O-MIMO configurations when the channel is known at the receiver and unknown at the transmitter as well as for MISO configurations when the channel is also available at the transmitter.

The ergodic capacity is defined as the ensemble average of the capacity for all of the channel realizations: $E[C]$. To obtain a closed-form expression of the ergodic capacity, we first expand (4) in Taylor series about the expected value of D ,

$$C = \mathbf{h} \log_2 E[D] + \mathbf{h} \log_2(e) \frac{D - E[D]}{E[D]} + \frac{-\mathbf{h} \log_2(e)}{2} \frac{(D - E[D])^2}{E[D]^2} + \dots \quad (6)$$

Applying the expectation operator to (6),

$$E[C] = \mathbf{h} \log_2 E[D] - \mathbf{h} \frac{\log_2(e)}{2} \frac{\mathbf{s}_D^2}{E[D]^2} + \dots, \quad (7)$$

where \mathbf{s}_D^2 is the variance of D . Considering (4) and (5), the first moment of D can be written as follows

$$E[D] = 1 + \frac{\mathbf{r}}{\mathbf{b}} n_R n_T. \quad (8)$$

Considering (4), (5) and (8), the variance of D will be given by

$$\mathbf{s}_D^2 = \frac{\mathbf{r}^2}{\mathbf{b}^2 \mathbf{a}^2} \text{var}[\|\mathbf{H}\|_F^2], \quad (9)$$

being $\text{var}[\cdot]$ the variance operator. Finally, considering (9), (8) and (5), expression (7) can be written as follows

$$E[C] = \mathbf{h} \log_2 \left(1 + \frac{\mathbf{r} n_R n_T}{\mathbf{b}} \right) - \frac{\mathbf{h} \log_2(e) \mathbf{r}^2 \text{var}[\|\mathbf{H}\|_F^2]}{2 \mathbf{a}^2 (\mathbf{b} + \mathbf{r} n_R n_T)^2} + \dots \quad (10)$$

The above expression constitutes a second-order approximation of the ergodic capacity as a function of the first two moments of the Frobenius norm of the channel matrix (vector in the MISO and SIMO cases). In appendix I we derive closed-form expressions of such moments as function of the multipath channel matrices \mathbf{V}_m .

IV. SIMULATION RESULTS

The expression (10) can be used to estimate the ergodic capacity of SIMO and MISO systems in a wide variety of fading channels.

A. MISO Rayleigh Channel

Fig. 1 compares the analytical prediction with the results obtained with Monte Carlo simulation for a MISO uncorrelated Rayleigh channel with four transmit antennas. The two curves correspond to the cases of channel known at the transmitter ($\mathbf{b} = 1$) and channel not known at the transmitter ($\mathbf{b} = n_T$). To use (10), the Rayleigh channel was modeled as a $M = 100$ waves multi-path channel, with channel vectors \mathbf{V}_m as it was mentioned in section II. The result shows that (10) is a tight approximation of the ergodic capacity in both cases and for any value of SNR.

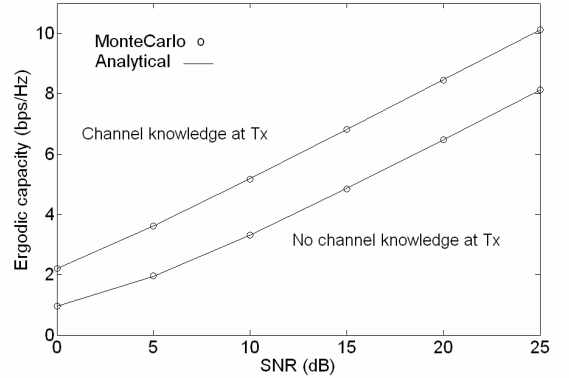


Figure 1. Comparison between the theoretical ergodic capacity and the second-order analytical prediction of (10) for an uncorrelated MISO Rayleigh channel with four transmit antennas.

B. SIMO correlated Rayleigh channel

Fig. 2 compares the analytical prediction with the results obtained with Monte Carlo simulation for a SIMO correlated Rayleigh channel with four receive antennas. The two curves correspond to correlation coefficients $\mathbf{r} = 0.5$ and $\mathbf{r} = 0.9$. To use (10), the Rayleigh channel was modeled with $M = 100$ multi-path waves, and the matrices \mathbf{V}_m were chosen according to (3). The result shows that (10) is a tight approximation of the ergodic capacity in both cases and for any value of SNR.

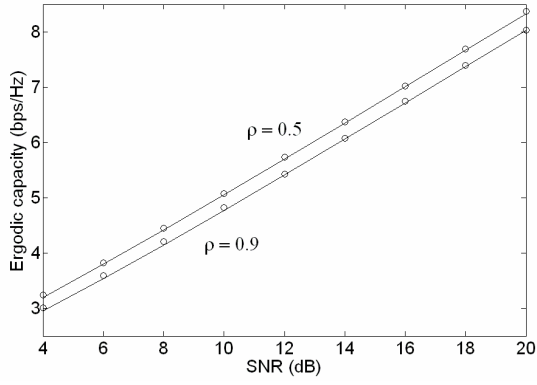


Figure 2. Comparison between the theoretical ergodic capacity and the second-order analytical prediction of (10) for a correlated SIMO Rayleigh channel with four receive antennas.

C. SIMO Ricean channel

Fig. 3 shows a comparison between the analytical prediction and Monte Carlo simulation for a Ricean SIMO channel considering different values of Ricean K-factor and different number of antennas. The average SNR is assumed to be 10 dB. To use (10), the multi-path channel was modeled as a $M=100$ waves channel with \mathbf{V}_m matrices as it was mentioned in section II. The power of the LOS component was adjusted according to the values of the desired K-factor. The figure shows that the analytical predictions fit quite well the ergodic capacity in any case.

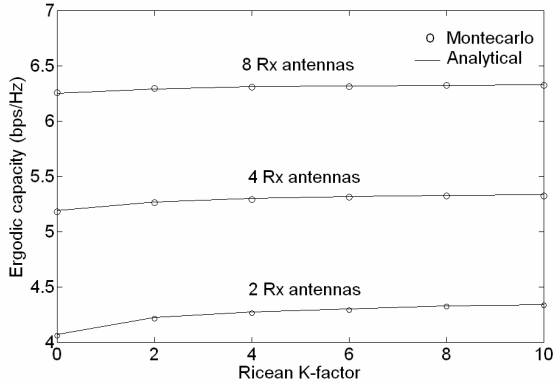


Figure 3. Comparison between the theoretical ergodic capacity and the second-order analytical prediction of (10) for a SIMO Ricean channel.

D. Urban microcellular scenario

To validate the analytical approximation in a variety of propagation conditions (channels), we have simulated the ergodic capacity of a spatial-diversity MISO system (assuming channel knowledge at the transmitter) in a specific microcellular environment depicted in fig. 4. It consists on 48 regularly distributed buildings of uniform height (25 m) and rectangular sections, forming a rectilinear grid of streets. This scenario represents an area of midtown Manhattan which has

been traditionally used for the validation of a number of outdoor propagation models. The area of the environment is 900 x 500 m. In the simulations we assume that the antennas are omni-directional and linear-vertical polarized. The transmitter and receiver are located 20m and 1.5m height, respectively. The transmitter array is assumed to be linear along the X axis in fig. 4, and the antennas spacing is 3 wavelengths. The simulations have been carried out at 1.8 GHz for different locations of the receiver along line-1 of fig. 4. To focus on the effect of multi-path fading only, we assume perfect power control, so the average path-loss always equals 1. In all the receiver locations the average signal-to-noise ratio is assumed to be 15 dB. The multi-path matrices \mathbf{V}_m have been calculated using a 3D ray-tracing propagation tool [11]. Fig. 5 compares the analytical predictions of (10) with Monte Carlo simulations.

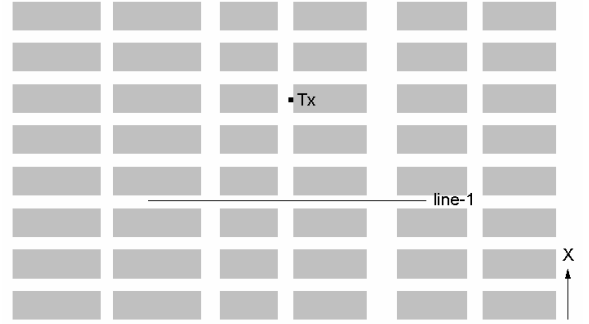


Figure 4. Top view of the urban microcell showing the transmitter array location. Line-1 shows the receiver antenna locations in the simulation.

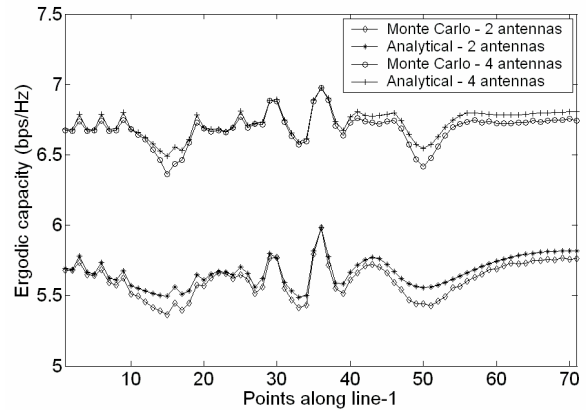


Figure 5. Simulated ergodic capacity along line-1 of fig. 1 for SNR=15 dB.

V. CONCLUSIONS

In this paper we have derived an analytical approximation of the ergodic capacity of MISO and SIMO systems, that is valid in a wide variety of fading channels. The approximation is valid for any SNR regime, any number of antennas and any correlation matrix. The expression is a closed-form function of

the channel parameters that directly depends on the arrays configurations, antennas characteristics and propagation environment.

APPENDIX I

From (1) the squared Frobenius norm of the channel can be expressed as follows

$$\|\mathbf{H}\|_F^2 = \text{Tr}(\mathbf{H}\mathbf{H}^H) = \sum_{m=1}^M \sum_{n=1}^M \text{Tr}(\mathbf{V}_m \mathbf{V}_n^H) \exp(j(\mathbf{f}_m - \mathbf{f}_n))$$

where $\text{Tr}(\cdot)$ denotes the trace of the corresponding matrix. Considering the statistical distributions of the multi-path phases \mathbf{f}_n , the moments of the squared Frobenius norm of the channel matrix can be expressed analytically as function of the channel matrices \mathbf{V}_m . For example, the mean will be

$$E[\|\mathbf{H}\|_F^2] = \sum_{m=1}^M \text{Tr}(\mathbf{V}_m \mathbf{V}_m^H) = \sum_{m=1}^M \|\mathbf{V}_m\|_F^2 \quad (11)$$

Similarly the fourth power of the Frobenius norm will be

$$\|\mathbf{H}\|_F^4 = \sum_{m_1}^M \sum_{n_1}^M \sum_{m_2}^M \sum_{n_2}^M \text{Tr}(\mathbf{V}_{m_1} \mathbf{V}_{n_1}^H) \text{Tr}(\mathbf{V}_{m_2} \mathbf{V}_{n_2}^H) \exp(j(\mathbf{f}_{m_1} - \mathbf{f}_{n_1} + \mathbf{f}_{m_2} - \mathbf{f}_{n_2}))$$

Then,

$$E[\|\mathbf{H}\|_F^4] = \sum_{m=1}^M \sum_{n=1}^M \|\mathbf{V}_m\|_F^2 \|\mathbf{V}_n\|_F^2 + \sum_{m=1}^M \sum_{\substack{n=1 \\ n \neq m}}^M |\text{Tr}(\mathbf{V}_m \mathbf{V}_n^H)|^2$$

Therefore, the variance of the squared Frobenius norm will be

$$\text{var}[\|\mathbf{H}\|_F^2] = \sum_{m=1}^M \sum_{\substack{n=1 \\ n \neq m}}^M |\text{Tr}(\mathbf{V}_m \mathbf{V}_n^H)|^2 \quad (10)$$

REFERENCES

- [1] I. E. Telatar, "Capacity of multi-antenna Gaussian channels," European Trans. Tel. vol. 8, pp. 585 - 595, November / December 1999.
- [2] O. Oyman, R. Nabar, H. Bolcskei, and A.J. Paulraj, "Tight lower bounds on the ergodic capacity of Rayleigh fading MIMO channels," Proc. IEEE GLOBECOM, Taipei, Taiwan, November 2002.
- [3] A. Grant, "Rayleigh fading multiantenna channels," EURASIP J. Applied Signal Processing, no. 3, pp. 316 - 329, March 2002.
- [4] E. Gauthier, A. Yongacoglu, J.Y. Chouinard, "Capacity of multiple antenna in Rayleigh fading channels," Canadian J. Electr. Comp. Eng. Vol. 25, pp. 105 - 108, July 2000.
- [5] C. Martin, B. Ottersten, "Analytical approximations of eigenvalue moments and mean channel capacity for mimo channels," Proc. IEEE ICASSP, 3, pp. 2389 - 2392, Orlando, FL, USA, May 2002.
- [6] C. Martin, B. Ottersten, "Analytical approximations of eigenvalue moments and mean channel capacity for mimo channels," Proc. IEEE ICASSP, 3, pp. 2389 - 2392, Orlando, FL, USA, May 2002.
- [7] L. Musavian, M. Dohler, M.R. Nakhai, and A.H. Aghvami, "Closed-form capacity expressions of orthogonalized correlated MIMO channels," IEEE Communication Letters, vol. 8, pp. 365-367, June 2004.

- [8] A. Paulraj, R. Nabar, and D. Gore, Introduction to Space-Time Wireless Communications. Cambridge University Press, 2003.
- [9] G. D. Durgin, Space-time Wireless Channels. Prentice Hall PTR, Upper Saddle River, NJ, 2003.
- [10] G. J. Foschini, and M. J. Gans, "On limits of wireless communications in a fading environment when using multiple antennas," Wireless Personal Communications, 1998, vol. 6, no. 3, pp. 311-335, March 1998.
- [11] M. F. C  drea, and J. P  rez, Cell Planning for Wireless Communications. Artech House, Norwood, MA, 1999

