

Some Weighted Objective Approaches For Sparse Deconvolution

Ignacio SANTAMARIA-CABALLERO[†]

Aníbal R. FIGUEIRAS-VIDAL[‡]

[†]*Dpto. Electrónica, ETSI Telecom-Univ. Cantabria Av. Los Castros s.n., 39005 Santander, SPAIN.*

[‡]*GPSS/DSSR, ETSI Telecom - UPM Ciudad Universitaria s.n., 28040 Madrid, SPAIN*

Tel: 341-336-7226, Fax:341-336-7350

e-mail: ignacio@gts.ssr.upm.es

Abstract. In this paper we present two new methods to perform sparse deconvolution. Basically, they use a gradient-type algorithm to minimize an objective function which consists on a weighted sum of the L_2 -norm of the residual and the L_1 -norm of the signal. Statistical information about the signal and noise can be efficiently used in our methods; for example, to derive a convergence criterion or to modify the weighted cost function. After discussing these methods, their effectiveness and robustness are illustrated by means of computer simulations using synthetic data.

1. Introduction

In many practical problems (geophysical signal processing, speech coding, synthetic aperture radar, nondestructive testing, etc.), it is often necessary to find a sparse solution to

$$\mathbf{z} = \mathbf{H}\mathbf{x} + \mathbf{n} \quad (1)$$

where \mathbf{H} is an impulse matrix, \mathbf{x} the unknown signal vector, \mathbf{n} the noise, and \mathbf{z} the observations. The signal \mathbf{x} is known to be sparse, i.e., it contains many zeroes plus several comparatively large nonzero values.

A direct application of standard methods to solve (1), such as using the Moore-Penrose pseudoinverse, is not appropriate, since the ill-conditioned character of the problem prevents the obtention of the sparse solutions we are looking for. In addition, theoretical solutions to the corresponding detection plus estimation problem are cumbersome; those available, such as [1], require that the peaks of \mathbf{x} have a Gaussian distribution, a hypothesis which is not acceptable in many situations. Moreover, sometimes signal statistics are not available, so a complete analytical solution is not possible.

To overcome these limitations, many alternatives have appeared in the literature: some of them [2] combine detection and estimation tasks using an adaptively contracted selection operator. This technique is computationally attractive, but is very sensitive to the selection of the parameters involved in the method and, more important, sometimes misses small peaks in the first steps of the detection process; an irreversible mistake.

An alternative approach consists on adding a regularizing term to reduce the ill-conditioned character of the problem, similar to the method proposed by Katsaggelos *et al.* [3], to restore noisy-blurred images. Nevertheless, it uses a constraint operator which produces a smooth solution not adequate for our problem.

Finally, other approaches based on using an L_1 -norm minimization algorithm have been proposed. In particular, the method presented in [4] performs the L_1 -norm mini-

mization of the residuals by means of linear programming. It was not proposed for its ability to recover sparse signals, but rather for its ability to deal with large data errors. A more appropriate approach consists on including the L_1 -norm of the signal in a weighted objective function [5,6]; this approach is well suited for data driven from spiky distributions. However, it has two drawbacks: the high computational cost when the data set is large, and the sparse character of the resultant residual, which does not agree with the usual type of added noise.

In this paper we propose some new methods to avoid the problems of the previously mentioned algorithms. Specifically, we present a new objective function that consists on a weighted sum of the L_2 -norm of the residual and the L_1 -norm of the signal. Some statistical information about the signal and noise can be efficiently used in this approach to stop the algorithm. Following similar ideas, another weighted function is proposed: it includes the *a priori* statistical knowledge in the objective function. In the rest of the paper we discuss these approaches and illustrate their performances by means of computer simulations using synthetic data.

2. Proposed Algorithm

The first method studied in this paper consists on finding the solution \mathbf{x} that minimizes the following cost function

$$\Phi_1(\hat{\mathbf{x}}, \alpha) = (1 - \alpha)\|\mathbf{z} - \mathbf{H}\hat{\mathbf{x}}\|_2 + \alpha\|\hat{\mathbf{x}}\|_1 \quad 0 \leq \alpha \leq 1 \quad (2)$$

By doing this, we seek a minimum L_1 -norm solution while preserving a small L_2 -norm residual. The weighting parameter α controls the spiky character of \mathbf{x} . The minimization of (2) can be accomplished by means of a gradient-type algorithm (with α fixed)

$$\hat{\mathbf{x}}_{k+1} = \hat{\mathbf{x}}_k + \mu(1 - \alpha)\mathbf{H}^T(\mathbf{z} - \mathbf{H}\hat{\mathbf{x}}_k) - \mu\alpha \text{sign}(\hat{\mathbf{x}}_k) \quad (3)$$

where the superscript T denotes transpose. The iterations are carried out until a previously specified error criterion

is satisfied or a maximum number of iterations is reached. Note that parameter α controls the sparse character of \mathbf{x} : for $\alpha=0$, the pseudoinverse solution of (1) is obtained and, as α is increased, spikier solutions are obtained; finally, when $\alpha=1$, the algorithm (3) converges to the trivial solution $\mathbf{x}=\mathbf{0}$.

On the other hand, for $\alpha=0$, μ must be less than twice the inverse of the maximum eigenvalue of $\mathbf{H}\mathbf{H}^T$ in order to guarantee convergence. In general, convergence depends on both μ and α . Since α is a bounded parameter ($0 \leq \alpha \leq 1$), we can choose an empirical μ to ensure convergence for any α .

A complete application of the method requires a procedure to select an optimum α which leads to a feasible solution. Without additional information we must choose α empirically. Nevertheless, as we will show later, α is not a critical parameter; so it can be fixed *a priori* for a given problem obtaining good results in a wide variety of examples. More accurate results can be obtained if we assume that some (very general) *a priori* knowledge about the original signal and noise is available (for example coming from a reasonable model). We will assume that estimates of the L_2 -norm of the noise (\hat{N}_2) and the L_1 -norm of the signal (\hat{S}_1) are available (the true values will be denoted without the symbol $\hat{\cdot}$). The estimate of the noise variance has been used in several signal deconvolution and image restoration problems [7,8] to derive convergence criteria, to choose the regularization parameter, or to find a projection operator onto a convex set such as $C_{\hat{N}_2} = \{\mathbf{x} \mid \|\mathbf{z} - \mathbf{H}\mathbf{x}\|_2 \leq \hat{N}_2\}$. In [8] it is noted that many other constraints can be applied to the deconvolution problem. They depend upon the characteristics of the specific signal; in particular, when we are dealing with a sparse signal, to use a similar constraint over the L_1 -norm of the signal has proven to be useful.

The proposed method starts selecting an α_{max} that produces a solution sparse enough. Then, the weighting parameter is iteratively decreased in fixed steps $\Delta\alpha$ until some convergence criterion is fulfilled. The criterion presented in this paper is based on the derivative of the objective function with respect to α : each α yields a solution \mathbf{x}^* which minimizes (2), the value of the objective function for this solution is $\Phi_1(\mathbf{x}^*, \alpha)$. Assuming that the solution \mathbf{x}^* does not depend on α , we can write

$$\frac{d\Phi_1(\mathbf{x}^*, \alpha)}{d\alpha} = -\|\mathbf{z} - \mathbf{H}\mathbf{x}^*\|_2 + \|\mathbf{x}^*\|_1 \quad (4)$$

when \mathbf{x}^* is similar to the true series, the term $\|\mathbf{z} - \mathbf{H}\mathbf{x}^*\|_2$ approaches the L_2 -norm of the noise and $\|\mathbf{x}^*\|_1$ the L_1 -norm of the signal. Consequently, our knowledge about (\hat{N}_2) and (\hat{S}_1) can be used to stop the algorithm when a near optimum α has been reached, i.e., we know in advance an estimate of the derivative (4) for the true signal

$$\frac{\widehat{d\Phi_1}(\mathbf{x}, \alpha)}{d\alpha} = \hat{S}_1 - \hat{N}_2 \quad (5)$$

thus, the final solution \mathbf{x}^* , and the final weighting parameter α_{opt} , must satisfy the following criterion

$$\left| \frac{d\Phi_1(\mathbf{x}^*, \alpha_{opt})}{d\alpha} - \frac{\widehat{d\Phi_1}(\mathbf{x}, \alpha)}{d\alpha} \right| < \delta \quad (6)$$

where δ is a small positive constant.

It is also possible to use simpler convergence criteria. For example, using only information about the L_1 -norm of the signal: $|\|\mathbf{x}^*\|_1 - \hat{S}_1| < \delta$, or the L_2 -norm of the noise: $|\|\mathbf{z} - \mathbf{H}\mathbf{x}^*\|_2 - \hat{N}_2| < \delta$. The three criteria provide similar performance, as it will be shown in Section 4. Summarizing, the proposed algorithm is as follows

- 1 Initialize $\mathbf{x}_0 = \mathbf{0}_{N \times 1}$, $j=0$, $\alpha_0 = \alpha_{max}$
- 2 Estimate $\frac{\widehat{d\Phi_1}(\mathbf{x}, \alpha)}{d\alpha} = \hat{S}_1 - \hat{N}_2$
- 3 for $k=0$ to $N-1$
 - 3.1 $\hat{\mathbf{x}}_{k+1} = \hat{\mathbf{x}}_k + \mu(1 - \alpha_j)\mathbf{H}^T(\mathbf{z} - \mathbf{H}\hat{\mathbf{x}}_k) - \mu\alpha_j \text{sign}(\hat{\mathbf{x}}_k)$
 - 3.2 if $|\Phi_1(\hat{\mathbf{x}}_{k+1}, \alpha_j) - \Phi_1(\hat{\mathbf{x}}_k, \alpha_j)| < \epsilon$ go to 4
- end
- 4 $\mathbf{x}^* = \hat{\mathbf{x}}_{k+1}$
- 5 $\frac{d\Phi_1(\mathbf{x}^*, \alpha)}{d\alpha} \big|_{\alpha_j} = -\|\mathbf{z} - \mathbf{H}\mathbf{x}^*\|_2 + \|\mathbf{x}^*\|_1$
- 6 if $\left| \frac{d\Phi_1(\mathbf{x}^*, \alpha)}{d\alpha} \big|_{\alpha_j} - \frac{\widehat{d\Phi_1}(\mathbf{x}, \alpha)}{d\alpha} \right| < \delta$ then stop
 - else
 - 6.1 $\alpha_{j+1} = \alpha_j - \Delta\alpha$
 - 6.1 $j=j+1$
 - 6.1 go to 3
- end

Finally, to obtain a fully sparse signal we must apply a threshold procedure. The method selected proceeds in three steps: first, a conservative threshold is applied; second, the amplitudes of the surviving peaks are adjusted in order to minimize the quadratic error (this step increases the magnitude of the true peaks and, conversely, it decreases the false ones); finally, a new threshold is applied that eliminates the smaller spikes. The selection of the final threshold depends on the problem. For instance, in a multipulse coding application it could be selected to obtain a fixed number of spikes in the final solution.

3. An Alternative Approach

To minimize the L_1 -norm of the signal tends to underestimate the amplitudes of the true spikes. This effect can be reduced if we include our knowledge about \hat{S}_1 and \hat{N}_2 in the objective function. For example, according to

$$\Phi_2(\hat{\mathbf{x}}, \alpha) = (1 - \alpha) \left(\|\mathbf{z} - \mathbf{H}\hat{\mathbf{x}}\|_2 - \hat{N}_2 \right)^p + \alpha \left(\|\hat{\mathbf{x}}\|_1 - \hat{S}_1 \right)^q \quad (7)$$

with $0 \leq \alpha \leq 1$ and $1 \leq p, q$. This second approach has an additional advantage: it avoids the need to obtain the optimum α , since in this case the objective function by itself forces the vector \mathbf{x} to be in a neighborhood of the true solution. Therefore, an empirical α can be fixed in advance achieving good results for a great number of examples. For instance, $\alpha=0.5$ is the most obvious selection, thus giving the same importance to each term in the objective function. Other possibility is to change successively between $\alpha=0$ and $\alpha=1$. In this way, an alternative adjustment between the residual L_2 -norm and the signal L_1 -norm can be made. However, the simulations indicate a superior performance of the former approach (α fixed). Therefore we will consider only this procedure.

In addition to choose a weighting parameter we must

select the values p, q of the objective function. Experimental evidence shows that $p=q=1$ provide the best results. The minimization of (7) is again accomplished by means of a gradient-type algorithm. As previously, for α fixed the parameter of the gradient algorithm μ controls the convergence. A conservative selection is needed with this approach to avoid convergence problems.

4. Simulation Results

We have selected two computer experiments with different sparse signals. The first uses a deterministic signal; the second uses randomly generated sparse signals according to a preestablished model. Specifically, we generate sparse signals with Gaussian or uniform amplitude distributions.

4.1. Experiment 1

The impulse response used in this example corresponds to the first 20 points of an ARMA filter having a zero at $z=0.6$ and two poles at $z=0.8\exp(\pm j5\pi/12)$. The test signal \mathbf{x} is a 110 points register having nonzero values at points $x(20)=8$, $x(25)=6.845$, $x(47)=-5.4$, $x(71)=4$ and $x(95)=-3.6$. The SNR used in this example is 4 dB, and is defined as the power of $\mathbf{H}\mathbf{x}$ with respect to the power of \mathbf{n} ; \mathbf{n} being a zero mean Gaussian white noise.

We compare the performance of the algorithms corresponding to:

A1) Cost function $\Phi_1(\hat{\mathbf{x}}, \alpha)$ with an optimum weighting parameter chosen to fulfill (6).

A2) Cost function $\Phi_2(\hat{\mathbf{x}}, \alpha)$ with a fixed weighting parameter.

A3) Using an adaptive threshold [2].

For the three methods we ensure coverage by selecting $\mu=0.1$. We apply the first method with $\alpha_0=0.8$, and then update this parameter in fixed steps of $\Delta\alpha=0.01$ until (6) is fulfilled. For each α , a maximum number of 50 iterations of (3) is carried out. For the second method we have empirically selected an optimum $\alpha=0.5$ and a maximum number of iterations of 200. With respect to the selection of α we must remark that it is not a critical parameter: values in a neighborhood of the optimum α give visually satisfactory solutions.

Figure 1 shows the result obtained for one simulation applying the first algorithm after thresholding.

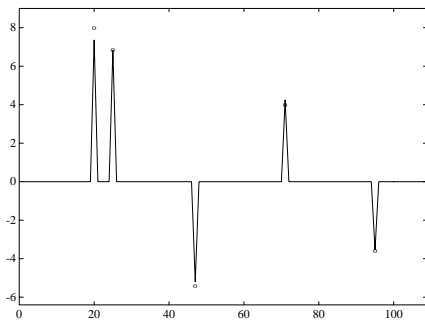


Figure 1. Solution obtained with the first algorithm after thresholding. Circles depict true spikes.

	A1	A2	A3
spike 1	100	100	100
spike 2	100	100	100
spike 3	99	99	96
spike 4	94	89	86
spike 5	84	82	67
average of spurious spikes	0.71	0.61	0.53

Table 1. Comparison of the results from the three algorithms tested in Example 1. The first row indicates the method used, the next five rows show the detection percentage for each spike, and the last row shows the averaged number of false peaks detected in each simulation.

Table 1 shows the averaged results for one hundred independent simulations. Both methods have similar detection capabilities and outperform the adaptive threshold procedure but increase the number of spurious spikes.

We have tested the first algorithm applying other convergence criteria. Using only the \hat{S}_1 estimate or the \hat{N}_2 estimate, all the criteria achieve similar results. This can be explained noting that if for a solution \mathbf{x}^* , $\|\mathbf{x}^*\|_1$ is close to S_1 , then, $\|\mathbf{z} - \mathbf{H}\mathbf{x}^*\|_2$ will be near to N_2 .

Both methods use estimates of N_2 and S_1 , therefore it is important to study their robustness against errors in the estimates. To do so, we have repeated the same one hundred simulations but in this case we introduced random errors in the estimates; i.e., we considered that \hat{S}_1 and \hat{N}_2 were two random variables uniformly distributed in the intervals $[S_1(1 - \epsilon_1), S_1(1 + \epsilon_1)]$ and $[N_2(1 - \epsilon_2), N_2(1 + \epsilon_2)]$ with $0 \leq \epsilon_1, \epsilon_2 \leq 1$. To simplify the results, we assume the same error percentage for both estimates, i.e., $\epsilon_1 = \epsilon_2 = \epsilon$. Figure 2 shows the worsening in detection percentage for the first algorithm and ϵ varying from 0 to 1. Besides, the average number of spurious spikes remains nearly constant. So we can conclude that the proposed method is remarkably robust. On the other hand, the minimization of $\Phi_2(\hat{\mathbf{x}}, \alpha)$ is less robust against errors in \hat{N}_2 and \hat{S}_1 . This is understandable because in the first approach the errors only affect the convergence criterion, while in the second approach they affect the objective function, thus modifying the algorithm itself.

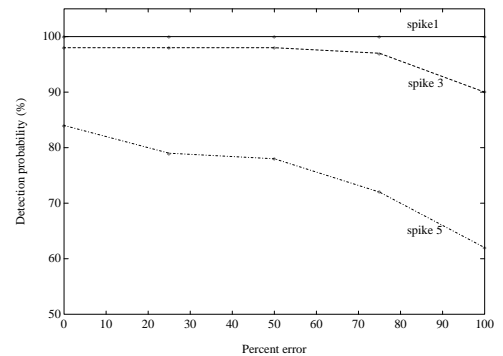


Figure 2. Degradation in the detection performance for the first method when there are errors in \hat{N}_2 and \hat{S}_1 .

4.2. Experiment 2

In this example we evaluate the performance of our algorithms using synthetic signals according to the following model: $x(k)=r(k)q(k)$; where $q(k)$ is a Bernoulli process for which $q(k)=1$ with probability λ and $q(k)=0$ with probability $1-\lambda$; $r(k)$ is a white random process with zero mean, variance σ_r^2 and whose amplitudes fit a Gaussian or uniform distribution (in particular, the Gaussian distribution is often used for seismic deconvolution cases). Registers of five hundred samples were generated according to the above models (with $\lambda = 0.05$ and $\sigma_r^2 = 10$), and then convolved with the source wavelet described in Example 1. Finally, a zero mean Gaussian noise was added to the result.

For this example, the simulations compare the performance of the algorithms corresponding to:

B1) Cost function $\Phi_1(\hat{x}, \alpha)$ with an optimum weighting parameter chosen to fulfill (6).

B2) Cost function $\Phi_2(\hat{x}, \alpha)$ with a fixed weighting parameter.

B3) One-shot threshold detector [1].

Figure 3 shows the results obtained with the first method before thresholding. The signal has a Gaussian amplitude distribution and the SNR= 8 dB.

Table 2 shows the averaged results of 25 simulations when there is a Gaussian (Table 2.a) or a uniform (Table 2.b) amplitude distribution. The SNR for this example is 4 dB. Somehow surprisingly, the three algorithms give better results for a uniform amplitude distribution of the sparse signal. However, this can be easily explained since for a fixed variance, data driven from a Gaussian distribution are near zero (and, therefore, are more difficult to detect) with higher probability than if they were driven from a uniform distribution. It is clear that the proposed methods outperform the one-shot threshold detector.

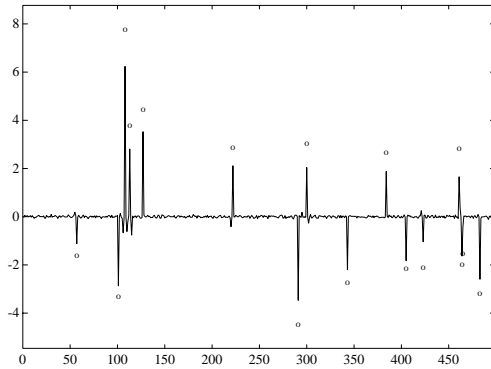


Figure 3. Solution obtained with the first algorithm before thresholding, Gaussian amplitude distribution and SNR= 8 dB. Circles depict true event amplitudes.

5. Conclusions

In this paper we have presented two new methods to recover a sparse signal from a noisy register. They use a gradient-type algorithm and incorporate statistical information about the signal and noise. The methods are computationally simple and efficient, and they achieve

a)	B1	B2	B3
correct detections (%)	70.7	68.1	56.6
false detections (%)	1.5	1.6	0.7

b)	B1	B2	B3
correct detections (%)	71.3	73.7	64.2
false detections (%)	1.5	1.3	0.4

Table 2. Averaged results for the three detectors (SNR=4 dB). The first column shows the average detection percentage, and the second the percentage of false peaks detected. a) Gaussian amplitude distribution and b) Uniform amplitude distribution.

good performance when applied to a wide variety of examples. The first method (which uses the statistical information to derive a convergence criterion to obtain the optimum weighting parameter) is more robust than the second (which includes the statistical information in the objective function), but it is slightly more involved.

Open study lines in this weighted objective approach are: to include other regularized functionals and other approaches to adjust the regularizing parameter; as well as to modify these methods to apply them to the (dual) problem of sinusoid detection.

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