AN EFFICIENT METHOD FOR CHAOTIC SIGNAL PARAMETER ESTIMATION

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ABSTRACT

Chaotic signals represent a new class of signals that can be applied in a wide range of signal processing applications. This paper describes an efficient method for parameter estimation of chaotic sequences generated by iterating tent maps with a known initial condition, and observed in white noise. The method is based on the connection between the symbolic sequence associated to the chaotic signal and the parameter of the tent map. The proposed method is asymptotically unbiased, and is found to attain the Cramer-Rao Lower Bound (CRLB) for a high Signal to Noise Ratio (SNR).

1 INTRODUCTION

Chaotic signals, signals generated by a non-linear dynamical system in chaotic state, have received much attention in the past years. Classical signal processing techniques are not adequate for this type of signals that, while deterministic, exhibit a noise-like behaviour. Consequently, it is important to develop new algorithms matched to this type of sequences. In particular, there is a need for robust and efficient algorithms for the estimation of these signals in noise.

Several papers address the problem of estimating the chaotic signal from noisy observations with known parameter [1][2][3]. In this case, signal estimation is equivalent to the estimation of the initial condition. Direct ML estimation of this initial condition is not possible because the likelihood function is highly irregular with fractal characteristics. However, a sequential ML estimator has been developed [3], as well as several suboptimal approaches [1][2]. Most of these papers use the strong connection between the sequence of signs of the signal elements (sometimes called itinerary) and the initial condition to develop estimators.

In this paper we address the parameter estimation problem: we observe a noisy chaotic signal $x[n]$ generated by a tent map with a known initial condition but unknown parameter. The direct ML estimation of the parameter is not possible because the likelihood function is also highly irregular, and the problem is solved mainly by linear techniques [4][5]. Chaotic modeling of the dynamics of a conversation [4], or communications using chaotic signals [5] are examples of applications where efficient parameter estimation is necessary.

In this paper we develop an efficient parameter estimation algorithm based on the symbolic dynamics characterization of the chaotic signal. The method is based on estimating the itinerary by hard censoring the chaotic signal, and calculating the set of parameter values that can produce such itinerary. As the number of data samples increases, this set becomes smaller, and a better estimation is obtained. The resulting estimator is asymptotically unbiased and attains the CRLB for a high SNR.

2 SYMBOLIC DYNAMICS OF TENT MAPS

The signals considered in this work $x[n]$ are generated by the non-linear map

$$x[n] = F(\beta, x[n-1]) \tag{1}$$

where $F(\cdot)$ is a symmetric tent map [6]

$$F(\beta, x) = \beta - 1 - \beta\,|x| \tag{2}$$

for some parameter $1 < \beta \leq 2$, and initial condition $x[0] \in [-1, \beta - 1]$. The function $F(\cdot)$ is non-invertible, but is also unimodal and even, so it has two inverse images that differ only in the sign. The two inverses of $F(\cdot)$ can be denoted

$$F_s^{-1}(v) = \frac{\beta - 1 - v}{\beta} s \tag{3}$$

where $s = \pm 1$. Therefore

$$v = F(x) \Rightarrow x = F^{-1}_{sign(x)}(v) \tag{4}$$

Thus, a useful representation of the chaotic sequence can be obtained via the inverse mapping (3), and any element of the sequence can be obtained by backward iteration from $x[N]$

$$x[n] = F^{-1}_{s[n]} \circ F^{-1}_{s[n+1]} \circ \cdots \circ F^{-1}_{s[N-1]}(x[N]) \tag{5}$$

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with $s[n] = sign(x[n])$. This sign sequence (also called itinerary) $\mathbf{s} = [s[0], ..., s[N]]$, can also be considered a symbolic coding of the chaotic signal [2]. Let us review briefly the symbolic characterization of sequences generated by tent maps. Let E_{-1}, E_1 be a partition of the phase space: a signal sample $x[n]$ belongs to E_{-1} if $s[n] < 0$ and to E_1 if $s[n] > 0$. Any orbit of the dynamical system can be encoded in a string $s[0], s[1], ..., s[n]...$ which is called symbolic sequence. Thus, we can write

$$F^n(\beta, x[0]) \in E_{s[n]} \Leftrightarrow x[0] \in F^{-n} E_{s[n]} \tag{6}$$

where F^n denotes the n-fold composition of F, and $F^{-n}E_{s[n]}$ is the set of points that map to the set $E_{s[n]}$ after n iterations. Suppose that we obtain a finite symbolic sequence $\mathbf{s}$ from a map with a known parameter β. Using $\mathbf{s}$ we can limit $x[0]$ to the set $Q_{\mathbf{s}} = \bigcap_{n=0}^{N} F^{-n}E_{s[n]}$. This set becomes increasingly smaller as N grows, thus providing an increasingly more accurate information about $x[0]$.

The methods for signal estimation presented in [1][2] are based on the symbolic dynamics characterization of $x[n]$.The itinerary is obtained by hard censoring the signal, and the initial condition is obtained either by backward iteration using (5) [2], or by using certain topological conditions [1]. It is assumed that, if the number of signal points is high enough, a random point of the region associated to the estimated itinerary is a good enough estimate for all practical purposes.

An alternative to those approaches is to calculate the regions $Q_{\mathbf{s}}$. In this case the region limits of the $Q_{\mathbf{s}}$ sets can be easily obtained by searching the zeros of $F^k(\beta, x[0])$, for example by backward iteration from zero [7]. In this paper we apply the same approach to the parameter estimation problem (i.e. we obtain the regions in the parameter range where a certain itinerary is produced).

3 PARAMETER ESTIMATION

The data model for the problem we are considering is

$$y[n] = x[n] + w[n] \qquad n = 0, 1, \ldots, N \tag{7}$$

where $w[n]$ is a stationary, zero-mean, white Gaussian noise, with variance σ^2, and $x[n]$ is generated using (2) by iterating some known $x[0] \in [-1, \beta - 1]$, according to (1), for some unknown parameter $1 < \beta \leq 2$.

Our approach to parameter estimation is based on the ideas presented in the previous section. Given a known initial condition $x[0]$ and an itinerary $\mathbf{s} = [s[0], ..., s[N]]$, only a subset of all the possible values of β produce the known itinerary starting from $x[0]$. That is

$$F^n(\beta, x[0]) \in E_{s[n]} \Leftrightarrow \beta \in F_{\beta}^{-n} E_{s[n]} \tag{8}$$

where $F_{\beta}^{-n}E_{s[n]}$ is the set of β values that map $x[0]$ to the set $E_{s[n]}$ after n iterations. Using $\mathbf{s}$ we can limit β to the set $R_{\mathbf{s}} = \bigcap_{n=1}^{N} F_{\beta}^{-n}E_{s[n]}$, which becomes increasingly smaller as N grows. Consequently, an increasingly better estimate of parameter β may be obtained.

To develop our estimator we need to calculate the region limits in the β domain. For that purpose the following lemma is necessary.

Lemma: Given a known initial condition $x[0]$, the set of parameter values $(\beta | F^k(\beta, x[0]) = 0)$, for $0 \leq k < n$, divide the range of β $(1, 2]$ in a finite number of subintervals (regions), inside which all the values of β share the same symbolic sequence.

Proof: Let β_0, β_1 be two points strictly inside one of these subintervals. Their itineraries must coincide. Otherwise, there is some $j < n$ such that, say $F^j(\beta_0, x[0]) < 0$ and $F^j(\beta_1, x[0]) > 0$. Then, by continuity, there must be a β_2 between β_0 and β_1 for which $F^j(\beta_2, x[0]) = 0$, a contradiction.$\square$

The problem of finding the zeros of F^k is much more complex in this case than in the case of the estimation of the initial condition, because F^k is a polynomial of order k in β with different coefficients for each itinerary, whereas it is linear in $x[0]$. In the first place, we have to obtain a closed form expression for $F^n(\beta, x[0])$ given a certain itinerary $\mathbf{s}$. It is easily shown by induction that F^n is given by

$$F^n_{s[0],\ldots,s[n-1]}(\beta, x[0]) = (\beta - 1) \sum_{i=0}^{n-1} S_i^n \beta^i + S_n^n \beta^n x[0] \tag{9}$$

where $S_0^n = 1$, and

$$S_i^n = (-1)^i \prod_{j=n-i}^{n-1} s[j] \qquad i = 1, \ldots, N \tag{10}$$

It is important to note that $F^n_{s[0],\ldots,s[n-1]}$ is only equivalent to $F^n(\beta, x[0])$ for values of β that produce the given itinerary with the known initial condition $x[0]$. The method to obtain the estimate of β directly from the roots of F^n is

1. $\boldsymbol{\beta} = [1, 2], \mathbf{S} = sign(x[0])$
2. FOR $n = 1, ..., N$.
   - $[\boldsymbol{\beta}, \mathbf{S}]$=FIND-ROOTS$(\boldsymbol{\beta}, \mathbf{S}, n)$;
     % $\boldsymbol{\beta}$ is a vector with the limits of the regions, and $\mathbf{S}$ is a matrix whose rows contain the itinerary associated to each region.

   END
3. Estimate the itinerary $\hat{\mathbf{s}} = [\hat{s}[1], \ldots, \hat{s}[N]]$ by hard censoring $y[1], \ldots, y[N]$.

   % The signs are obtained directly from the noisy sequence: $\hat{s}[n] = sign(y[n])$
4. IF $\hat{\mathbf{s}}$ is not valid ($\hat{\mathbf{s}}$ is not equal to any row of $\mathbf{S}$). THEN Modify the sign sequence (as explained below)

5. Estimate β from the set of values that comply with the itinerary

The estimate of β is obtained either as an average, a random value, or by conducting a fine search in the previous set if more precision is required. The algorithm FIND-ROOTS, which is the main part of the method, can be summarized as follows

1. FOR i=1,...,length($\boldsymbol{\beta}$)-1
   - Construct $F^n_{\mathbf{S}(i,1),\ldots,\mathbf{S}(i,n)}$
   - Obtain the roots of $F^n_{\mathbf{S}(i,1),\ldots,\mathbf{S}(i,n)}$
   - Select the roots that belong to $[\boldsymbol{\beta}(i), \boldsymbol{\beta}(i+1)]$, and add them to $\boldsymbol{\beta}$

   END
2. Construct the new sign matrix $\mathbf{S}$

Obviously, in a practical implementation the limits of the regions are obtained once and stored in a look-up table, performing only steps 3-5 of the algorithm subsequently.

Finally, when the estimated itinerary $\hat{\mathbf{s}}$ is not valid, the sign sequence is modified as follows. In the first place, the point k of the sequence where $|y[n]|$ is minimum is found, and a new sign sequence $\hat{s}[1], ..., -\hat{s}[k], ..., \hat{s}[N]$ constructed and tested. If this new sign sequence is also found not to be valid, a second point of the sequence where $|y[n]|$ is minimum r ($r \neq k$) is again selected, and all the possible combinations (except the ones known not to be valid) are examined. In this case there would be two possible combinations

$$\begin{aligned}\hat{\mathbf{s}}_1 &= \hat{s}[1], ..., -\hat{s}[r], ..., -\hat{s}[k], ..., \hat{s}[N] \\ \hat{\mathbf{s}}_2 &= \hat{s}[1], ..., -\hat{s}[r], ..., \hat{s}[k], ..., \hat{s}[N]\end{aligned}$$

that would have to be tested. If none of them is found to be valid the search for a valid itinerary continues in the same way until a valid sign sequence is found.

4 AN EFFICIENT METHOD

The method to obtain the roots in the previous algorithm is quite costly computationally, and may not be feasible at all for high N. This complexity is highly reduced by assuming the convexity of the regions (i.e. monotonicity of the function) associated to each itinerary. With this hypothesis, only one root of F^n inside each of the regions associated to F^{n-1} has to be found. The proposed method proceeds sequentially searching the unique possible zero-crossing of F^n inside each region of F^{n-1} to obtain the regions associated to F^n. For $n = 1$ there is only one value (or none) that makes $F(\beta, x[0]) = 0$

$$\beta^* = 1/(1 - |x[0]|) \qquad (11)$$

For $\beta > \beta^*$ the sign of $x[1]$ is positive, while for $\beta < \beta^*$ it is negative. This divides the range of β in two regions. To proceed further we need to find the roots of $F^2(\beta, x[0])$ in each of the regions previously obtained. At most there can be one zero in each region, so we can evaluate the function in the two region limits: if the resulting signs are opposite there is a zero-crossing, otherwise no zero-crossing is present. In the first case, any zero finding algorithm, such as Newton-Raphson, may be used to find the new region limits. In the second case, only one sign is possible for the next step, and consequently our symbolic sequence may not be admissible, forcing us to modify it as explained previously. Thus, the algorithm to estimate β is the following

1. $\boldsymbol{\beta} = [1, 2], \mathbf{S} = sign(x[0])$
2. FOR $n = 1, ..., N$.
   - FOR i=1,...,length($\boldsymbol{\beta}$)-1
     - Obtain $l_0 = F^n(\boldsymbol{\beta}(i), x[0])$, and $l_1 = F^n(\boldsymbol{\beta}(i+1), x[0])$
       % Evaluate $F^n(\beta, x[0])$ at the borders of the present region
     - IF $sign(l_0) \neq sign(l_1)$ THEN Apply Newton-Raphson to find β^*, and add it to $\boldsymbol{\beta}$
       % Obtain the zero of $F^n(\beta, x[0])$ in the range $[\boldsymbol{\beta}(i), \boldsymbol{\beta}(i+1)]$

     END
   - Construct the new sign matrix $\mathbf{S}$

   END
3. Estimate the itinerary $\hat{\mathbf{s}} = [\hat{s}[1], \ldots, \hat{s}[N]]$ by hard censoring $y[1], \ldots, y[N]$.

   % The signs are obtained directly from the noisy sequence: $\hat{s}[n] = sign(y[n])$
4. IF $\hat{\mathbf{s}}$ is not valid ($\hat{\mathbf{s}}$ is not equal to any row of $\mathbf{S}$). THEN Modify the sign sequence (as explained previously)
5. Select the region that complies with the itinerary, and estimate β as the middle point of the selected region

Once again, in a practical implementation the regions are calculated once and stored in a look-up table, performing only steps 3-5 of the algorithm subsequently.

5 COMPUTER SIMULATIONS

We compare by Monte Carlo simulations the performance of the proposed methods with a fine grid search (by minimizing the sum of the error squares directly) and the Total Least-Squares (TLS) solution. The fine grid search is implemented in a similar way to [1]: a coarse search is conducted by dividing the interval $[1, 2]$

into 1000 points. Then we search over the interval $[\hat{\beta}_{min} - 0.001, \hat{\beta}_{min} + 0.001]$ using 1000 points, where $\hat{\beta}_{min}$ is the coarsely obtained β. Both proposed methods are equivalent, because they produce the same partition of the parameter range. The estimate of β is obtained performing a fine search over the selected region using 1000 points. We use a data record of length $N+1$ with $N = 10$, $x[0] = -0.61$, and $\beta = 1.45$

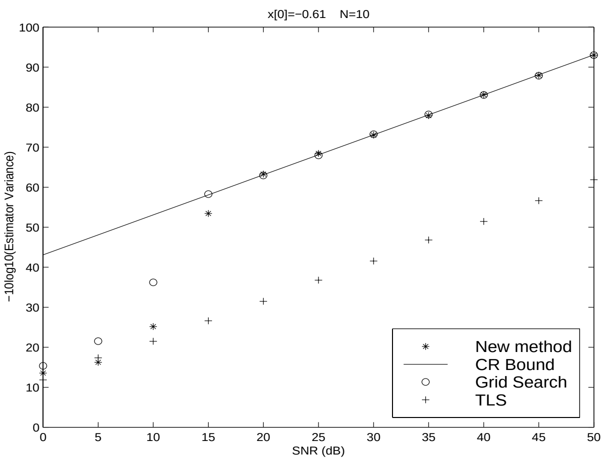

Figure 1: Performance of the different estimators ($\beta = 1.45$).

The results are shown in Fig. 1. The proposed methods are asymptotically unbiased and achieve the CRLB for about 18 dB. They improve considerably the TLS solution and their performace is comparable to the fine grid search (at a much lower computational cost).

In Fig. 2 we plot the proposed estimators variance as a function of the number of data points N for three different SNR levels, 12, 15 and 18 dB. 10000 cases have been averaged for each SNR and register length, with initial condition $x[0] = -0.63$ and $\beta = 1.9$. A threshold effect can be appreciated: the MSE does not decrease when N increases beyond a certain register length. This behaviour is similar to the one observed in the estimation of the initial condition [2], and suggests that the information is lost after a number of iterations.

6 CONCLUSIONS

In this paper we have presented an efficient method for estimating the parameter of chaotic signals generated by iterating tent maps and observed in white noise when the initial condition is known. The method is based on estimating the itinerary by hard censoring the data, and calculating the set of parameter values that can produce such itinerary. As the number of data samples increases, this set becomes smaller. Therefore, the middle point of such a region can be a reasonable estimate when the number of points is high enough. Otherwise, a fine search in the selected region may be conducted. The proposed estimators are asymptotically unbiased, and achieve the CRLB for a high SNR. Future research lines include the development of ML estimators for chaotic signal parameters and the application of these estimators to chaos based communications systems.

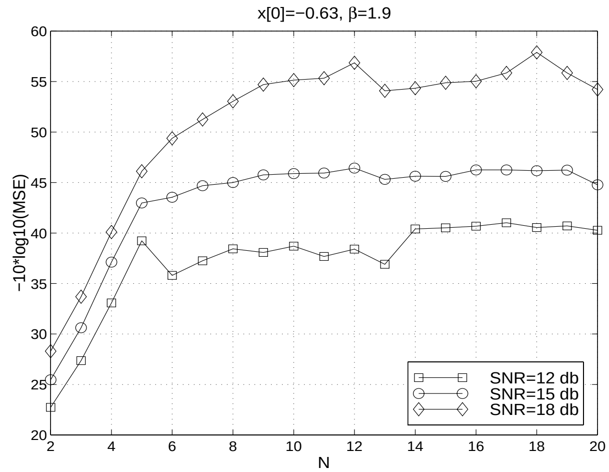

Figure 2: MSE of the estimates as a function of the register length for different SNRs. ($\beta = 1.9$).

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