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Signal Processing 83 (2003) 659–664

SIGNAL PROCESSING

www.elsevier.com/locate/sigpro

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Bayesian estimation of chaotic signals generated by piecewise-linear maps☆

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Received 30 March 2001

Abstract

Chaotic signals are potentially attractive in a wide range of signal processing applications. This paper deals with Bayesian estimation of chaotic sequences generated by piecewise-linear (PWL) maps and observed in white Gaussian noise. The existence of invariant distributions associated with these sequences makes the development of Bayesian estimators quite natural. Both maximum a posteriori (MAP) and minimum mean square error (MS) estimators are derived. Computer simulations confirm the expected performance of both approaches, and show how the inclusion of a priori information produces in most cases an increase in performance over the maximum likelihood (ML) case.
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PACS: 05.45.+b

Keywords: Bayesian estimation; Chaos; Piecewise linear maps

1. Introduction

Chaotic signals, i.e., signals generated by a non-linear dynamical system in chaotic state, represent an attractive class of signals for modeling physical phenomena. Furthermore these signals may provide a new mechanism for signal design in communications and remote sensing applications. From a signal processing perspective, the detection, estimation, analysis and characterization of this type of signals represents a significant challenge [5]. In this work, we deal with chaotic signals generated by iterating one-dimensional PWL maps. These maps, although very simple, show a very complex behaviour, and have been applied in several signal processing and communication applications. An important class of PWL maps are the Markov maps. Signals generated by Markov maps have many interesting properties, for example, all of them have rational power spectral densities [3]. In this work, we consider the estimation of signals generated by PWL maps and observed in white Gaussian noise.

Several authors have proposed signal estimation algorithms for chaotic signals [1,4,10]. These methods are usually based on the connection between the symbolic sequence associated to a particular chaotic signal and its initial condition, and are, in general, suboptimal. A dynamical programming approach has been

☆ This work has been partially supported by the European Commission under Grant 1FD97-1066-C02-01 and Spanish Ministry of Science and Technology under project TIC2001-0751-C04-03.

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0165-1684/03/$ - see front matter © 2002 Elsevier Science B.V. All rights reserved.
doi:10.1016/S0165-1684(02)00485-1

also proposed [4]. ML estimators have been developed for the tent map dynamics [8], and generalized to all PWL maps [7]. In [6] a Bayesian estimator for chaotic signals generated by the tent map has been proposed. A recent issue of the Proceedings of IEEE has been devoted to applications of chaotic signals to electronic and information engineering; with many interesting applications of chaotic signals to communications, modeling and signal processing [2].

In this paper, we extend the study of Pantaleón et al. [6] and develop Bayesian estimators for chaotic signals whose dynamics are governed by PWL maps. The selection of the a priori density is based on the invariant density associated with chaotic sequences generated by PWL maps. The derivation of the Bayesian estimators is based on a closed form expression for the n-fold composition of the PWL map, necessary to obtain the posterior density. The resulting MAP and MS estimators show good performance at low SNR in comparison with ML estimates, are asymptotically unbiased, and achieve the CRLB at high SNR.

2. Symbolic dynamics of piecewise-linear maps

In this section, we discuss briefly the symbolic dynamics of PWL maps, a more detailed discussion can be found in [7,11]. We consider PWL maps $F:[0,1] \to [0,1]$. The interval $[0,1]$ is partitioned into M disjoint convex intervals E_i, $i=1,\dots,M$. Then F is defined as

$$F(x) = a_i x + b_i \quad \text{if } x \in E_i, \tag{1}$$

where all the a_i and b_i are known constants. This formulation includes all piecewise linear maps on the unit interval, continuous or not, including Markov maps [3]. An example of Markov map is shown in Fig. 1; it is defined by the coefficients

$$a_1 = \frac{1-a}{a}, \quad b_1 = a, \quad a_2 = \frac{-1}{1-a},$$

$$b_2 = \frac{1}{1-a} \tag{2}$$

with $E_1=[0,a]$ and $E_2=[a,1]$, that satisfy the Markovian property that partition points map to partition points. Chaotic signals may be generated by iterating an unknown initial condition $x[0] \in [0,1]$ according to

$$x[n] = F(x[n-1]). \tag{3}$$

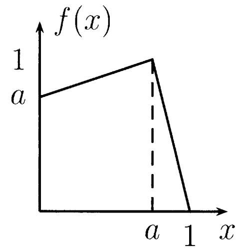

Fig. 1. PWL Markov map with two partition elements.

We will denote F^k the k-fold composition of F. We will also denote the symbolic sequence—itinerary—associated to a chaotic signal of length $N+1$ as a length N sequence of symbols $\mathbf{s} = \{s[0], s[1], \dots, s[N-1]\}$, where $s[k]=i$ if $F^k(x[0]) \in E_i$. We define S_N as the map that associates an initial condition in $[0,1]$ to its corresponding symbolic sequence of length N.

We can define another partition of the phase space in a collection of P intervals R_j composed of the points in $[0,1]$ that share a common length N symbolic sequence. Given a certain itinerary $\mathbf{s}_j$ we define $R_j = \{x \in [0,1]: S_N(x) = \mathbf{s}_j\}$. Every point in the phase space belongs to one and only one of these sets, and, if the E_i's are convex sets, the R_j's are also convex [11]. If all the linear components are onto, all the symbolic sequences are admissible and there are $P = M^N$ regions R_j. If any of the linear components is not onto, some symbolic sequences will not be admissible and $P < M^N$.

We will denote $F^k_{\mathbf{s}}(x[0])$ as the k-fold composition of F for an initial condition $x[0]$ with $S_N(x[0]) = \mathbf{s}$. Given a known itinerary, we can write a closed form expression for $F^k_{\mathbf{s}}(x[0])$. If we define

$$A^{l,k}_{\mathbf{s}} = \prod_{n=k-l}^{k-1} a_{s[n]} \tag{4}$$

and $A^{0,k}_{\mathbf{s}} = 1$, then we have

$$F^k_{\mathbf{s}}(x[0]) = A^{k,k}_{\mathbf{s}} x[0] + \sum_{l=0}^{k-1} A^{l,k}_{\mathbf{s}} b_{s[k-1-l]}. \tag{5}$$

Note the linear dependence on the initial condition in (5). The index $\mathbf{s}$ stresses the fact that (5) is only equivalent to $F^k(x[0])$ if the itinerary of $x[0]$ is given by $\mathbf{s}$. Let us finally define an indicator (sometimes called characteristic) function

$$\chi_j(x) = \begin{cases} 1 & \text{if } x \in R_j, \\ 0 & \text{if } x \notin R_j. \end{cases} \tag{6}$$

With this definition we may express $F^k(x)$ as

$$F^k(x) = \sum_{j=1}^{P} \chi_j(x) F^k_{\mathbf{s}_j}(x). \tag{7}$$

As a conclusion, given any PWL map, we can define a partition of the phase space in convex intervals R_j in which all the points share the same symbolic sequence of length N. In this case, given an itinerary $\mathbf{s}_j$, (5) is a closed form expression for $F^k(x[0])$ in the domain R_j.

3. Bayesian estimation of PWL map sequences

3.1. Problem statement

The data model for the problem we are considering is

$$y[n] = x[n] + w[n], \qquad n = 0, 1, \ldots, N, \tag{8}$$

where $w[n]$ is a stationary, zero-mean white Gaussian noise with variance σ^2, and $x[n]$ is generated using (1) by iterating some unknown $x[0]$ according to (3). In this paper, we will address Bayesian estimation of the initial condition $x[0]$. The rest of the signal components may be estimated in a similar way.

3.2. Prior density

To develop the Bayesian estimators, we need to define the prior density for the initial condition $x[0]$. The obvious choice is to assign the invariant density associated with the known map. In many cases, closed form expressions exist for these densities, as is the case of Markov maps [3] where they are piecewise-constant. In those cases for which no closed form expressions are available, we propose to express the prior densities according to

$$p(x[0]) = \sum_{j=1}^{P} p_j \chi_j(x[0]), \tag{9}$$

where the p_j constants are given by

$$p_j = \frac{1}{\Delta_j} \int_{R_j} p(x)\,\mathrm{d}x, \tag{10}$$

and Δ_j is the width of the region R_j. The invariant density is substituted by a staircase approximation. This model is exact in many cases, as in Markov maps, and is a reasonable approximation in most cases. For example the map in (2) has an invariant density with the form of (9) with $p_1 = 1/(1+a)$ and $p_2 = 1/(1-a^2)$ [3]. In a general case, the constants p_j may be estimated from long sequences of data generated according to the model given by (1) and (3).

3.3. Posterior density

Since our observations $\mathbf{y} = \{y[0], y[1], \ldots, y[N]\}$ are a collection of independent Gaussian random variables with equal variance, the conditional density may be expressed as

$$p(\mathbf{y}|x[0]) = \frac{1}{(\sqrt{2\pi}\sigma)^{N+1}} \exp\left(-\frac{J(x[0])}{2\sigma^2}\right),$$

where $J(x[0])$ is given by

$$J(x[0]) = \sum_{k=0}^{N} (y[k] - F^k(x[0]))^2. \tag{11}$$

Using (5), we can express (11) in a certain region R_j as

$$J_j(x[0]) = \sum_{k=0}^{N} (y[k] - F^k_{\mathbf{s}_j}(x[0]))^2, \tag{12}$$

so we can write (11) as

$$J(x[0]) = \sum_{j=1}^{P} \chi_j(x[0]) J_j(x[0]).$$

Finally, the conditional density may be expressed as

$$p(\mathbf{y}|x[0]) = \frac{1}{(\sqrt{2\pi}\sigma)^{N+1}} \sum_{j=1}^{P} \chi_j(x[0]) \times \exp\left(-\frac{J_j(x[0])}{2\sigma^2}\right).$$

Applying the Bayes rule, and using (9) as the prior density, the posterior density becomes

$$p(x[0]|\mathbf{y}) = K \sum_{j=1}^{P} p_j \chi_j(x[0]) \exp\left(-\frac{J_j(x[0])}{2\sigma^2}\right), \tag{13}$$

where K is a normalization constant. It is clear from (5) that (12) is a quadratic function of $x[0]$ in each R_j. Differentiating and solving for the unique minimum we obtain

$$\hat{x}_j[0] = \frac{\sum_{k=0}^{N}(y[k] - \sum_{l=0}^{k-1} A_{\mathbf{s}_j}^{l,k} b_{s_j[k-1-l]})A_{\mathbf{s}_j}^{k,k}}{\sum_{k=0}^{N}(A_{\mathbf{s}_j}^{k,k})^2}, \tag{14}$$

where $A_{\mathbf{s}_j}^{l,k}$ is given by (4) with itinerary $\mathbf{s}_j$ and $s_j[n]$ is the nth component of $\mathbf{s}_j$. Using (14) we can express $J_j(x[0])$ in the following way

$$J_j(x[0]) = J_j(\hat{x}_j[0]) + (x[0] - \hat{x}_j[0])^2 \sum_{k=0}^{N}(A_{\mathbf{s}_j}^{k,k})^2. \tag{15}$$

Finally, substituting (15) into (13), and after some straightforward calculations, we obtain

$$p(x[0]|\mathbf{y}) = K \sum_{j=1}^{P} q_j \chi_j(x[0]) \times \exp\left(-\frac{(x[0] - \hat{x}_j[0])^2}{2\sigma_j^2}\right), \tag{16}$$

where

$$q_j = p_j \exp\left(-\frac{J_j(\hat{x}_j[0])}{2\sigma^2}\right),$$

$$\sigma_j^2 = \sigma^2 \left(\sum_{k=0}^{N}(A_{\mathbf{s}_j}^{k,k})^2\right)^{-1}$$

and

$$K^{-1} = \sum_{j=1}^{P} p_j B_j, \tag{17}$$

where

$$B_j = \int_{R_j} \exp\left(-\frac{(x[0] - \hat{x}_j[0])^2}{2\sigma_j^2}\right) \mathrm{d}x[0].$$

3.4. Bayesian estimators

The posterior density given by (16) is composed of P truncated Gaussians weighted by the coefficients q_j, as is shown in Fig. 2 for the map defined by (2) for $N = 3$, $x[0] = 0.4$ and SNR = 10 dB. The maximum

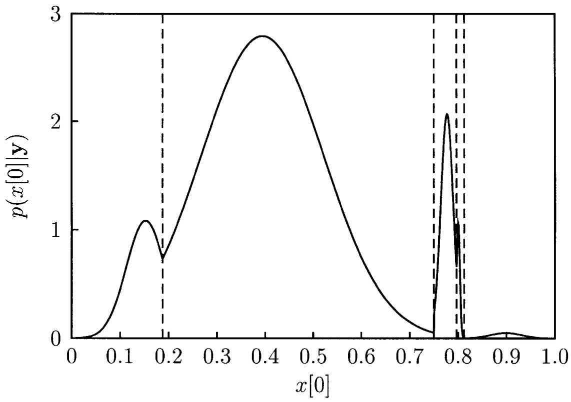

Fig. 2. Posterior density of the observations for $N = 3$, $x[0] = 0.4$ and SNR = 10 dB. The dashed lines mark the region boundaries.

of each Gaussian is given by the ML estimate for a known itinerary [7]

$$\hat{x}_{\mathrm{ML}}^{j}[0] = \min(R_j^{\mathrm{u}}, \max(R_j^{\mathrm{l}}, \hat{x}_j[0])), \tag{18}$$

where R_j^{l} and R_j^{u} are the lower and upper limits of the region R_j.

The MAP estimator is obtained from the ML estimator as $\hat{x}_{\mathrm{MAP}}[0] = \hat{x}_{\mathrm{ML}}^{m}[0]$, where

$$m = \arg\max_j \left\{ p_j \exp\left(-\frac{J(\hat{x}_{\mathrm{ML}}^{j}[0]}{2\sigma^2}\right)\right\}, \quad j = 1, \ldots, P. \tag{19}$$

The MS estimate is the mean of the posterior density given by (16). The resulting estimator is

$$\hat{x}_{\mathrm{MS}}[0] = K \sum_{j=1}^{P} q_j B_j \left(\hat{x}_j[0] - \frac{\sigma_j^2}{B_j} g(\hat{\xi}_j)\right), \tag{20}$$

where $\hat{\xi}_j = \hat{x}_j[0] - d_j$ is the displacement of each estimate from the midpoint, d_j, of the associated region of width Δ_j, and

$$g(\hat{\xi}_j) = \exp\left(-\frac{(\hat{\xi}_j - \Delta_j/2)^2}{2\sigma_j^2}\right) - \exp\left(-\frac{(\hat{\xi}_j + \Delta_j/2)^2}{2\sigma_j^2}\right). \tag{21}$$

Both Bayesian estimators clearly resemble, in each region, the well-known MAP and MS estimators for a constant signal in Gaussian noise with uniform prior density [9].

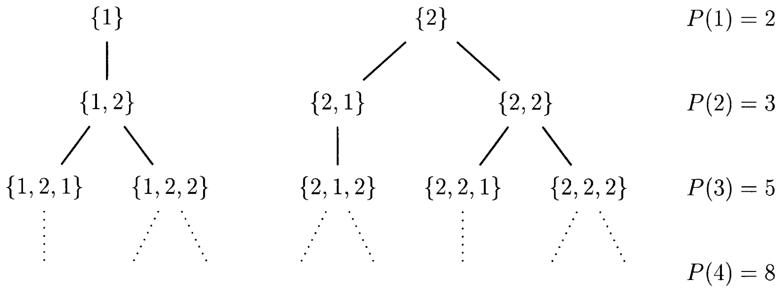

Fig. 3. Number of symbolic sequences of length N.

4. Numerical results

In this section, we return to the Markov map defined in (2). For this PWL map not all the itineraries are admissible. In particular, since all $x \in E_1$ are mapped onto E_2, it is not possible for any itinerary to have $s[n]=s[n+1]=1$. Taking into account this restriction, it is straightforward to derive a recurrence rule for $P(N)$, the number of itineraries of length N. As shown in Fig. 3, the number of symbolic sequences of length N can be decomposed into two groups. On the one hand, for each itinerary of length $N-1$, a length N itinerary can be obtained by appending a symbol 2 at the end. On the other hand, for each itinerary of length $N-1$ that ends with a symbol 2, a length N itinerary can be obtained by appending a symbol 1 at the end. However, according to the rule for the first group, the number of length $N-1$ itineraries that ends with symbol 2 must be equal to the total number of itineraries of length $N-2$. From the construction above, it is clear that $P(N)=P(N-1)+P(N-2)$, the same recurrence rule as the Fibonacci sequence. Given the initial conditions $P(1)=2$ and $P(2)=3$, the recurrence can be solved to provide the closed form

$$P(N) = \frac{5+3\sqrt{5}}{10}\left(\frac{1+\sqrt{5}}{2}\right)^N + \frac{5-3\sqrt{5}}{10}\left(\frac{1-\sqrt{5}}{2}\right)^N.$$

The number of symbolic sequences, which determines the number of regions in the partition of the phase space, grows much slower than the limit 2^N. For example, for $N=10$, the number of regions is 144, almost an order of magnitude smaller than the limit 1024.

To check the method, a PWL given by (2) with $a=0.75$ has been chosen. In all the simulations the register length is $N=6$. One thousand initial conditions $x[0]$ have been randomly generated according to the prior density of the map. For each one of the initial conditions, the sequence $x[n]$ is generated by iterating (3). This sequence is used to build 1000 noisy sequences $y[n]$ for each SNR. Table 1 shows the MSE for the three estimators.

Table 1
MSE of the three estimators as a function of the SNR

SNR	MSE (dB)		
	ML	MAP	MS
10	18.2	20.7	20.8
15	30.1	31.1	33.0
20	40.0	40.6	41.9
25	47.9	48.2	49.4
30	54.9	55.1	55.9
45	73.1	73.2	73.3

An example of the behaviour of the three estimators for a given initial condition is shown in Fig. 4 also with $a=0.75$. In this case, (3) has been used to generate a sequence $x[n]$ for the initial condition $x[0]=0.83$. From this sequence, one thousand noisy signals $y[n]$ have been randomly generated for each SNR, and the MSE has been calculated by averaging the results.

5. Conclusions

In this work, we have developed Bayesian estimators for a class of chaotic signals generated by

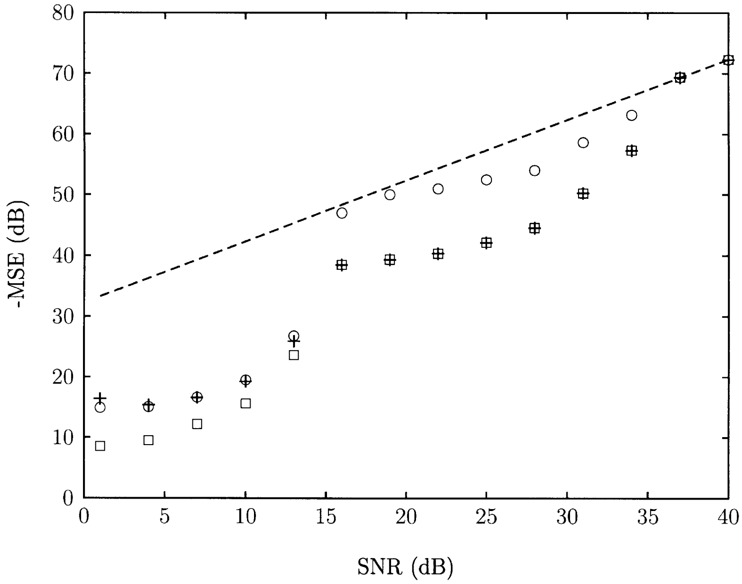

Fig. 4. The mean square error (MSE) for the three estimators: ML (□), MAP (+), and MS (○) is shown against the Cramer–Rao lower bound (dashed line).

iterating PWL maps and observed in white Gaussian noise. Using the itinerary associated to any chaotic sequence, we have developed a closed form expression for the conditional density of the initial condition. Applying the Bayes rule, we obtain the posterior density using the invariant density of the map as prior density. The resulting Bayesian estimators improve the ML estimator performance at low SNRs. The computational cost, although increased over the ML estimator, remains reasonable for moderate sized data records.

Further lines of research include searching for new performance bounds that better capture the performance of optimal estimators and looking for suboptimal estimation approaches that reduce the computational cost by only considering a small subset of the possible itineraries.

References

[1] L. Cong, W. Xiafou, S. Songgeng, A general efficient method for chaotic signal estimation, IEEE Trans. Signal Process. 47 (5) (1999) 1424–1428.

[2] M. Hasler, G. Maxxini, M. Ogorzatek, R. Rovatti, G. Setti (Eds.), Applications of nonlinear dynamics to electronic and information engineering (special issue), Proc. IEEE 90 (5) 2002.

[3] S.H. Isabelle, G.W. Wornell, Statistical analysis and spectral estimation techniques for one-dimensional chaotic signals, IEEE Trans. Signal Process. 45 (6) (1997) 1495–1506.

[4] S.M. Kay, V. Nagesha, Methods for chaotic signal estimation, IEEE Trans. Signal Process. 43 (8) (1995) 2013–2016.

[5] A.V. Oppenheim, K.M. Cuomo, Chaotic signals and signal processing, in: V.K. Madiseti, D.B. Williams (Eds.), The Digital Signal Processing Handbook, CRC Press, Boca Raton, FL, 1998.

[6] C. Pantaleón, D. Luengo, I. Santamaría, Bayesian estimation of a class of chaotic signals, Proceedings of the 2000 IEEE International Conference on Acoustics, Speech, and Signal Processing (ICASSP 2000), Vol. I, Istanbul, Turkey, June 2000, pp. 197–200.

[7] C. Pantaleón, D. Luengo, I. Santamaría, Optimal estimation of chaotic signals generated by piecewise-linear maps, IEEE Signal Process. Lett. 7 (8) (August 2000) 235–237.

[8] H.C. Papadopoulos, G.W. Wornell, Maximum likelihood estimation of a class of chaotic signals, IEEE Trans. Inform. Theory 41 (1) (1995) 312–317.

[9] M. Schwartz, L. Shaw, Signal Processing: Discrete Spectral Analysis, Detection and Estimation, McGraw-Hill, New York, 1975.

[10] S. Wang, P.C. Yip, H. Leung, Estimating initial conditions of noisy chaotic signals generated by piecewise-linear Markov maps using itineraries, IEEE Trans. Signal Process. 47 (12) (1999) 3289–3302.

[11] C.W. Wu, L.O. Chua, Symbolic dynamics of piecewise-linear maps, IEEE Trans. CAS II: Analog Digital Signal Process. 41 (6) (1994) 420–424.